Every biregular function is a biholomorphic map

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WCAA 2007
Workshop on Clifford Analysis and Applications
Ghent 2007

Outline

Fueter-regular functions

- $\mathbb{H} \simeq \mathbb{C}^2$:

$$\mathbb{C}^2 \ni z = (z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$$
$$\longleftrightarrow q = z_1 + z_2 j = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H}$$

- Ω bounded domain in $\mathbb{H}$. A quaternionic function $f = f_1 + f_2 j \in C^1(\Omega)$ is (left) regular (or hyperholomorphic) on Ω if it is in the kernel of the Cauchy-Riemann-Fueter operator

$$\mathcal{D} = 2\left(\frac{\partial}{\partial \bar{z}_1} + j\frac{\partial}{\partial \bar{z}_2}\right) = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3} \quad \text{on } \Omega$$

(cf. Nōno 1985, Shapiro and Vasilevski 1995, ...)

1. The identity function is regular
2. The space $\mathcal{R}(\Omega)$ of regular functions on Ω is a *right* $\mathbb{H}$-module with integral representation formulas
3. Every holomorphic map (f_1, f_2) on Ω defines a regular function $f = f_1 + f_2 j$

Fueter-regular functions

$f = f_1 + f_2 j$ is regular on Ω if and only if the Jacobian matrix

$$J(f) = \left(\frac{\partial(f_1, f_2, \bar{f}_1, \bar{f}_2)}{\partial(z_1, z_2, \bar{z}_1, \bar{z}_2)} \right)$$

is a regular matrix at every $z \in \Omega$, of the form

$$J(f) = \left(\begin{array}{cc|cc} a_1 & -\bar{b}_2 & -\bar{c}_2 & -c_1 \\ a_2 & \bar{b}_1 & \bar{c}_1 & -c_2 \\ \hline -c_2 & -\bar{c}_1 & \bar{a}_1 & -b_2 \\ c_1 & -\bar{c}_2 & \bar{a}_2 & b_1 \end{array} \right)$$

where $a = \left(\frac{\partial f_1}{\partial z_1}, \frac{\partial f_2}{\partial z_1} \right)$, $b = \left(\frac{\partial \bar{f}_2}{\partial \bar{z}_2}, -\frac{\partial \bar{f}_1}{\partial \bar{z}_2} \right)$,
$c = \left(\frac{\partial \bar{f}_2}{\partial z_1}, -\frac{\partial \bar{f}_1}{\partial z_1} \right) = -\left(\frac{\partial f_1}{\partial \bar{z}_2}, \frac{\partial f_2}{\partial \bar{z}_2} \right)$.

Biregular functions

A quaternionic function $f \in C^1(\Omega)$ is called biregular if

$$f \text{ is invertible and } f, f^{-1} \text{ are regular}$$

If this property holds locally, f is called locally biregular (cf. Królikowski and Porter 1994 and Królikowski 1996)

1. The class $\mathcal{BR}(\Omega)$ of biregular functions is closed respect to right multiplication by $a \in \mathbb{H}^*$, but it is not closed respect to composition or sum: even if $f+g$ is invertible, $f, g \in \mathcal{BR}(\Omega)$, the sum can be not biregular

Example

$f = 2\bar{z}_1 + 2\bar{z}_2 j$, $g = z_1 + (z_1 + z_2)j$ are biregular, the sum $f+g$ is invertible and regular but not biregular

2. Every biholomorphic map (f_1, f_2) on Ω defines a biregular function $f = f_1 + f_2 j$

Biregular functions

Examples

1. The identity function is biregular on $\mathbb{H}$
2. More generally, the affine functions $f(q) = qa + b$, $a \in \mathbb{H}^*$, $b \in \mathbb{H}$, are biregular on $\mathbb{H}$
3. $f = \bar{z}_1 + \bar{z}_2 j \in \mathcal{R}(\mathbb{H})$, $f^{-1} = f \in \mathcal{BR}(\mathbb{H})$
4. The function $f = z_1 + z_2 + \bar{z}_1 + (z_1 + z_2 + \bar{z}_2)j$ is regular, but

$$f^{-1} = \frac{1}{3}(z_1 + z_2 + \bar{z}_1 - 2\bar{z}_2 + (z_1 + z_2 - 2\bar{z}_1 + \bar{z}_2)j)$$

   is not regular.

Remark: In example 4, $\det J(f) = -3 < 0$.

Biregular functions

A regular function f is locally biregular if and only if $\det J(f) \neq 0$ at $z \in \Omega$ and $J(f^{-1})$ is a regular matrix at $f(z)$. Equivalently,

$$\det J(f) \neq 0 \quad \text{and } f \text{ satisfies the nonlinear differential system}$$

$$\begin{cases} e_1 := (a_2c_1 - a_1c_2)\bar{a}_1 + (-b_2c_1 + b_1c_2)\bar{b}_1 + (-a_2b_1 + a_1b_2)\bar{c}_1 = 0 \\ e_2 := (a_2c_1 - a_1c_2)\bar{a}_2 + (-b_2c_1 + b_1c_2)\bar{b}_2 + (-a_2b_1 + a_1b_2)\bar{c}_2 = 0 \end{cases}$$

where $a = \left(\frac{\partial f_1}{\partial z_1}, \frac{\partial f_2}{\partial z_1}\right)$, $b = \left(\frac{\partial \bar{f}_2}{\partial \bar{z}_2}, -\frac{\partial \bar{f}_1}{\partial \bar{z}_2}\right)$, $c = \left(\frac{\partial \bar{f}_2}{\partial z_1}, -\frac{\partial \bar{f}_1}{\partial z_1}\right)$.

Hypercomplex structure on $\mathbb{H}$

- Hypercomplex structure on $\mathbb{H} \simeq \mathbb{C}^2$: J_1, J_2 complex structures on $T\mathbb{H} \simeq \mathbb{H}$ defined by left multiplication by i and j
- J_1^*, J_2^* dual structures on $T^*\mathbb{H}$. We make the choice $J_3^* = J_1^* J_2^* \Rightarrow J_3 = -J_1 J_2$
- We can rewrite the equations of regularity as

$$df + iJ_1^*(df) + jJ_2^*(df) + kJ_3^*(df) = 0 \quad \text{(Joyce 1998)}$$

or, in complex components f_1, f_2,

$$\overline{\partial} f_1 = J_2^*(\partial \overline{f}_2)$$

Holomorphic functions w.r.t. a complex structure J_p

Let $J_p = p_1 J_1 + p_2 J_2 + p_3 J_3$ be the orthogonal complex structure on $\mathbb{H}$ defined by a unit imaginary quaternion $p = p_1 i + p_2 j + p_3 k$ in the sphere $\mathbb{S}^2$. Every J_p-holomorphic function $f = f^0 + if^1 : \Omega \to \mathbb{C}$ i.e.

$$df^0 = J_p^*(df^1) \quad \Leftrightarrow \quad df + iJ_p^*(df) = 0$$

defines a regular function $\tilde{f} = f^0 + pf^1$ on Ω.
We can identify $\tilde{f}$ with a holomorphic function

$$\tilde{f} : (\Omega, J_p) \to (\mathbb{H}, L_p)$$

where L_p is the complex structure defined by left multiplication by p.
(Note that $L_p = J_{p'}$, where $p' = p_1 i + p_2 j - p_3 k$)

Holomorphic maps w.r.t. a complex structure J_p

Space of holomorphic maps from (Ω, J_p) to $(\mathbb{H}, L_p)$

$$Hol_p(\Omega, \mathbb{H}) = \{f : \Omega \to \mathbb{H} \mid \overline{\partial}_p f = 0 \text{ on } \Omega\} = Ker \overline{\partial}_p$$

(J_p-holomorphic maps on Ω) where $\overline{\partial}_p$ is the Cauchy-Riemann operator w.r.t. J_p:

$$\overline{\partial}_p = \frac{1}{2}\left(d + pJ_p^* \circ d\right).$$

For any positive orthonormal basis $\{1, p, q, pq\}$ of $\mathbb{H}$ ($p, q \in \mathbb{S}^2$), the equations of regularity can be rewritten in complex form as

$$\overline{\partial}_p f_1 = J_q^*(\partial_p \overline{f}_2)$$

where $f = (f^0 + pf^1) + (f^2 + pf^3)q = f_1 + f_2 q$

$\Rightarrow$ every $f \in Hol_p(\Omega, \mathbb{H})$ is a regular function on Ω.

Holomorphic functions w.r.t. non-constant almost complex structures

If $p = p(z) \in \mathbb{S}^2$ varies smoothly with $z \in \Omega$, we get a similar result. Every $J_{p(z)}$-holomorphic map $f : (\Omega, J_{p(z)}) \to (\mathbb{H}, L_{p(f(z))})$ is regular:

$$\overline{\partial}_{p(z)} f = \frac{1}{2}\left[df(z) + p(f(z)) J^*_{p(f(z))} \circ df(z)\right] = 0 \quad \Rightarrow \quad f \in \mathcal{R}(\Omega)$$

$$\begin{aligned} \overline{\partial}_{p(z)} f = 0 \quad &\Rightarrow \quad \text{the linear map } df(z) \in Hol_{p(z)}(\Omega, \mathbb{H}) \\ &\qquad \text{for every fixed } z \in \Omega \\ &\Rightarrow \quad df(z) \in \mathcal{R}(\Omega) \ \text{ for every } z \in \Omega \\ &\Rightarrow \quad f \in \mathcal{R}(\Omega) \end{aligned}$$

Holomorphic functions w.r.t. non-constant almost complex structures

Example

$f(z) = \bar{z}_1 + z_2^2 + \bar{z}_2 j$ is regular on $\mathbb{H}$. On $\Omega = \mathbb{H} \setminus \{z_2 = 0\}$ f is holomorphic w.r.t. the almost complex structure $J_{p(z)}$, where

$$p(z) = \frac{1}{\sqrt{|z_2|^2 + |z_2|^4}} \left(|z_2|^2 i - (\mathrm{Im}\, z_2) j - (\mathrm{Re}\, z_2) k \right)$$

Also $f^{-1}(z) = \bar{z}_1 - z_2^2 + \bar{z}_2 j$ is regular on $\mathbb{H} \Rightarrow f$ is biregular on $\mathbb{H}$

Remark

f is biholomorphic: $f(\Omega) = \Omega$ and $f^{-1} \in Hol_{p'(f(z))}(\Omega, \mathbb{H}) \subset \mathcal{R}(\Omega)$, where

$$p'(f(z)) = \frac{1}{\sqrt{|z_2|^2 + |z_2|^4}} \left(|z_2|^2 i + (\mathrm{Im}\, z_2) j + (\mathrm{Re}\, z_2) k \right)$$

A criterion for holomorphicity

The energy density of a map $f : \Omega \to \mathbb{H}$, of class $C^1(\Omega)$, is

$$\mathcal{E}(f) = \frac{1}{2}\|df\|^2 = \frac{1}{2}\,\mathrm{tr}(J(f)\overline{J(f)}^T)$$

The energy of $f \in C^1(\overline{\Omega})$ on Ω is the integral

$$\mathcal{E}_\Omega(f) = \frac{1}{2}\int_\Omega E(f)dV$$

Theorem

If $f \in C^1(\overline{\Omega})$ is regular on Ω, then it minimizes energy in its homotopy class (relative to $\partial\Omega$).

(cf. Lichnerowicz 1970, Chen and Li 2000)

A criterion for holomorphicity

Let $A = (a_{\alpha\beta})$ be the 3×3 matrix with entries the real functions

$$a_{\alpha\beta} = -\langle J_\alpha, f^* L_{i_\beta} \rangle, \text{ where } (i_1, i_2, i_3) = (i, j, k).$$

For $f \in C^1(\overline{\Omega})$, we set

$$A_\Omega = \int_\Omega A \, dV \quad \text{and} \quad M_\Omega = \frac{1}{2}((\operatorname{tr} A_\Omega) I_3 - A_\Omega)$$

(Lichnerowicz invariants)

Theorem (P. 2005)

1. *f is regular on Ω if and only if $\mathcal{E}_\Omega(f) = \operatorname{tr} M_\Omega$.*
2. *If $f \in \mathcal{R}(\Omega)$, then M_Ω is symmetric and positive semidefinite.*
3. *If $f \in \mathcal{R}(\Omega)$, then f belongs to some space $Hol_p(\Omega, \mathbb{H})$ (for a constant structure J_p) if and only if $\det M_\Omega = 0$:*
   $X_p = (p_1, p_2, p_3)$ is a unit vector in the kernel of M_Ω if and only if $f \in Hol_p(\Omega, \mathbb{H})$.

A criterion for holomorphicity

The criterion holds also pointwise: let Ω be connected and $f \in C^1(\Omega)$. Consider the matrix of real functions on Ω

$$M = \frac{1}{2}\left((\operatorname{tr} A) I_3 - A\right)$$

Theorem

1. *f is regular on Ω if and only if $\mathcal{E}(f) = \operatorname{tr} M$ at every point $z \in \Omega$.*
2. *If $f \in \mathcal{R}(\Omega)$, then M is symmetric and positive semidefinite.*
3. *If $f \in \mathcal{R}(\Omega)$, then $\det M = 0$ on Ω if and only if there exists an open, dense subset $\Omega' \subseteq \Omega$ such that f belongs to $Hol_{p(z)}(\Omega', \mathbb{H})$ for some $p(z)$.*

Remark

If f is (real) affine, M is a constant matrix.
If f is not affine, $\det M = 0$ on Ω does not imply that $\det M_\Omega = 0$, but the converse is true.

A criterion for holomorphicity: examples

Linear examples

- $f = \bar{z}_1 + z_2 + \bar{z}_2 j$ is J_p-holomorphic, with $p = \frac{1}{\sqrt{5}}(i - 2k)$, since

$$\mathcal{E}_B(f) = \mathcal{E}(f) = 3 \quad \text{and} \quad M_B = M = \begin{bmatrix} 2 & 0 & 1 \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & \frac{1}{2} \end{bmatrix} \quad (Vol(B) = 1)$$

  f is biregular on $\mathbb{H}$, since it is biholomorphic:
  $f^{-1} = \bar{z}_1 - z_2 + \bar{z}_2 j$ is $J_{p'}$-holomorphic, with $p' = \frac{1}{\sqrt{5}}(i + 2k)$

- $f = z_1 + z_2 + \bar{z}_1 + (z_1 + z_2 + \bar{z}_2)j$ is regular, but not holomorphic:

$$\mathcal{E}(f) = 6 \quad \text{and} \quad M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

  Here $e_1 = 0$, $e_2 = 4 \Rightarrow f$ is not biregular.

A criterion for holomorphicity: examples

Linear examples

- $f = \bar{z}_1 + \bar{z}_2 j$ has matrix

$$M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

  of rank 1. This means that $f \in Hol_j(\mathbb{H}, \mathbb{H}) \cap Hol_k(\mathbb{H}, \mathbb{H})$. $f = f^{-1}$ is biholomorphic $\Rightarrow f \in \mathcal{BR}(\mathbb{H})$.
- $f = id \in Hol_i(\mathbb{H}, \mathbb{H}) \cap Hol_j(\mathbb{H}, \mathbb{H}) = \bigcap_{p \in <i,j>} Hol_p(\mathbb{H}, \mathbb{H})$.

A criterion for holomorphicity: examples

Examples (Nonlinear case)

- $f(z) = \bar{z}_1 + z_2^2 + \bar{z}_2 j$ is regular (also biregular) on $\mathbb{H}$:

$$\mathcal{E}(f) = 2 + 4|z_2|^2, \quad M = 2\begin{bmatrix} 1 & \operatorname{Im} z_2 & \operatorname{Re} z_2 \\ \operatorname{Im} z_2 & |z_2|^2 & 0 \\ \operatorname{Re} z_2 & 0 & |z_2|^2 \end{bmatrix} \Rightarrow \det M = 0$$

On $\Omega' = \mathbb{H} \setminus \{z_2 = 0\}$, where $\operatorname{rank} M = 2$, f is $J_{p(z)}$-holomorphic, with $p(z) = \frac{1}{\sqrt{|z_2|^2+|z_2|^4}} \left(|z_2|^2 i - (\operatorname{Im} z_2) j - (\operatorname{Re} z_2) k\right)$.

On the unit ball B, $\mathcal{E}_B(f) = \frac{10}{3}$ and $M_B = \int_B M dV = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{3} \end{bmatrix}$.

Since $\det M_B \neq 0$, f is not J_q-holomorphic for any constant complex structure J_q.

A criterion for holomorphicity: examples

Examples (Nonlinear case)

- $f = |z_1|^2 - |z_2|^2 + \bar{z}_1 \bar{z}_2 j$ has energy density $3|z|^2$. The matrices M_B and M are

$$M_B = \begin{bmatrix} \frac{4}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}, \quad M = \begin{bmatrix} 2|z|^2 & 0 & 0 \\ 0 & \frac{1}{2}|z|^2 & 0 \\ 0 & 0 & \frac{1}{2}|z|^2 \end{bmatrix}$$

$\Longrightarrow$ f is regular but not holomorphic w.r.t. any complex structure J_p. Note that $\det M = \frac{1}{2}|z|^6$ vanishes only at the origin.

Biregular functions are biholomorphic

Theorem

Let $f \in \mathcal{R}(\Omega)$, $M \geq 0$ be the real, symmetric matrix associated with f.

- *The following formula holds:*

$$\det M = \frac{1}{2}\left(|e_1|^2 + |e_2|^2\right) \quad \textit{where}$$

$$\begin{cases} e_1 = (a_2c_1 - a_1c_2)\bar{a}_1 + (-b_2c_1 + b_1c_2)\bar{b}_1 + (-a_2b_1 + a_1b_2)\bar{c}_1 \\ e_2 = (a_2c_1 - a_1c_2)\bar{a}_2 + (-b_2c_1 + b_1c_2)\bar{b}_2 + (-a_2b_1 + a_1b_2)\bar{c}_2 \end{cases}$$

- *If $f \in \mathcal{BR}(\Omega) \Rightarrow \exists \Omega' \subseteq \Omega$ open, dense subset and an (almost) complex structure $p(z)$ on Ω' such that $f \in Hol_{p(z)}(\Omega', \mathbb{H})$. Then $f : (\Omega', J_{p(z)}) \longrightarrow (f(\Omega'), L_{p(f(z))})$ is a biholomorphic map, with inverse $f^{-1} \in Hol_{p'(f((z))}(f(\Omega'), \mathbb{H})$.*

Biregular functions are biholomorphic

Remark

If f is locally biregular on Ω, then $e_1 = e_2 = 0 \Rightarrow \det M = 0$ on Ω. Then f is a local biholomorphism on an open, dense subset $\Omega' \subseteq \Omega$.

Corollary

If f is locally biregular on Ω, then $\det J(f) > 0$ on Ω. In particular, any such map f preserves orientation.

Sketch of proof

For any $f \in C^1(\Omega)$, if $d = -\left(\frac{\partial f_1}{\partial \bar{z}_2}, \frac{\partial f_2}{\partial \bar{z}_2}\right)$, we get

$$\mathcal{E}(f) = |a|^2 + |b|^2 + |c|^2 + |d|^2,$$

$$M = \begin{bmatrix} |c|^2 + |d|^2 & \mathrm{Im}(\langle a, d\rangle - \langle b, c\rangle) & \mathrm{Re}(\langle a, d\rangle + \langle b, c\rangle) \\ \mathrm{Im}(\langle a, c\rangle - \langle b, d\rangle) & \frac{1}{2}|a-b|^2 + \frac{1}{2}|c-d|^2 & \mathrm{Im}(\langle a, b\rangle + \langle c, d\rangle) \\ \mathrm{Re}(\langle a, c\rangle + \langle b, d\rangle) & \mathrm{Im}(\langle a, b\rangle - \langle c, d\rangle) & \frac{1}{2}|a+b|^2 + \frac{1}{2}|c-d|^2 \end{bmatrix}$$

Then $\mathcal{E}(f) = \mathrm{tr}\, M \Leftrightarrow c = d$, i.e. f is regular. In this case the matrix M becomes

$$M = \begin{bmatrix} 2|c|^2 & \mathrm{Im}\langle a-b, c\rangle & \mathrm{Re}\langle a+b, c\rangle \\ \mathrm{Im}\langle a-b, c\rangle & \frac{1}{2}|a-b|^2 & \mathrm{Im}\langle a, b\rangle \\ \mathrm{Re}\langle a+b, c\rangle & \mathrm{Im}\langle a, b\rangle & \frac{1}{2}|a+b|^2 \end{bmatrix}$$

Sketch of proof

If $f \in \mathcal{R}(\Omega)$, then $\mathcal{E}(f) = \mathrm{tr}\, M = \mathrm{tr}\, A$. Let

$$\mathcal{I}_p(f) = \frac{1}{2}\|df + L_{p\circ f} \circ df \circ J_p\|^2.$$

Then we obtain, as in Chen and Li 2000

$$\mathcal{E}(f) + \langle J_p, f^* L_{p\circ f}\rangle = \frac{1}{4}\mathcal{I}_p(f).$$

If $X = (p_1, p_2, p_3)$, then

$$XAX^T = \sum_{\alpha,\beta} p_\alpha p_\beta a_{\alpha\beta} = -\langle \sum_\alpha p_\alpha J_\alpha, f^* \sum_\beta p_\beta L_{i_\beta}\rangle$$

$$= -\langle J_p, f^* L_{p\circ f}\rangle = \mathcal{E}(f) - \frac{1}{4}\mathcal{I}_p(f).$$

Then $\mathrm{tr}\, A = \mathcal{E}(f) = XAX^T + \frac{1}{4}\mathcal{I}_p(f) \geq XAX^T$, with equality if and only if $\mathcal{I}_p(f) = 0$ i.e. if and only if f is a J_p-holomorphic map.

References

- Chen, J. and Li, J., *Quaternionic maps between hyperkähler manifolds*, J. Differential Geom. **55** (2000) 355–384
- Joyce, D., *Hypercomplex algebraic geometry*, Quart. J. Math. Oxford **49** (1998) 129–162
- W. Królikowski and R.M. Porter, *Regular and biregular functions in the sense of Fueter–some problems*, Ann. Polon. Math. **59** (1994) 53–64
- W. Królikowski, *On Fueter–Hurwitz regular mappings*, Diss. Math. **353** (1996) 1–91
- A. Lichnerowicz, *Applicationes harmoniques et variétés Kählériennes*, Symp. Math. **III**, Bologna, (1970) 341—402
- K. Nōno, *α-hyperholomorphic function theory*, Bull. Fukuoka Univ. Ed. III **35** (1985) 11–17
- P., *Holomorphic functions and regular quaternionic functions on the hyperkähler space* $\mathbb{H}$, Proceedings V ISAAC Congress, Catania 2005
- M.V. Shapiro and N.L. Vasilevski, *Quaternionic ψ-hyperholomorphic functions, singular integral operators and boundary value problems. I. ψ-hyperholomorphic function theory*, Complex Variables Theory Appl. **27** no.1 (1995) 17–46