On directional Hilbert operators for regular quaternionic functions on $\mathbb{R}^3$

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Abstract. In this paper we define directional quaternionic Hilbert operators on the three–dimensional space $\mathbb{H}_0 = \langle i, j, k \rangle \cong \mathbb{R}^3$. We consider functions in the kernel of the Cauchy-Riemann operator

$$\mathcal{D} = 2\left(\frac{\partial}{\partial \bar{z}_1} + j\frac{\partial}{\partial \bar{z}_2}\right) = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3},$$

a variant of the Cauchy–Fueter operator. This choice is motivated by the strict relation existing between this type of regularity and holomorphicity w.r.t. the whole class of complex structures on $\mathbb{H}$. For every imaginary unit $p \in \mathbb{S}^2$, let J_p be the corresponding complex structure on $\mathbb{H}$. Given a domain $\Omega \subseteq \mathbb{H}$, every holomorphic map from (Ω, J_p) to $(\mathbb{H}, L_p)$, where L_p is defined by left multiplication by p, is a regular function. We combine the quaternionic Cayley transformation, that maps the unit ball to the right half–space $\mathbb{H}^+ = \{q \in \mathbb{H} \mid Re(q) > 0\}$ with the Hilbert operators introduced in [16] on the unit sphere S of $\mathbb{H}$ in order to define directional Hilbert operators for (boundary values of) regular functions on $\mathbb{H}_0 \cong \mathbb{R}^3$.

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1. Introduction

The classical Hilbert transform expresses one of the real components of the boundary values of a holomorphic function in terms of the other. We are interested in a quaternionic analogue of this relation, which links the boundary values of one of the complex components of a regular function $f = f_1 + f_2 j$ $(f_1, f_2$ complex functions) to those of the other.

In [10] and [18] some generalizations of the Hilbert transform to hyperholomorphic functions were proposed. In these papers the functions considered are defined on plane

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or spatial domains, while we are interested in domains of two complex variables. In the latter case, pseudoconvexity becomes relevant, since a domain in $\mathbb{C}^2$ is pseudoconvex if and only if every complex harmonic function on it is a complex component of a regular function (cf. [11] and [12]).

In the complex variable case, there is a close connection between harmonic conjugates and the Hilbert transform, given by harmonic extension and boundary restriction. Several generalizations of this relation to higher dimensional spaces have been given (cf. e.g. [2, 3, 4, 6]), mainly in the framework of Clifford analysis, which can be considered as a generalization of quaternionic (and complex) analysis.

In [16] was introduced another variant of the quaternionic Hilbert operator, in which the (constant) complex structures on $\mathbb{H}$ play a decisive role. Since these structures depend on a "direction" p in the unit sphere $\mathbb{S}^2$ (cf. §2.2), this operator was called *directional Hilbert operator* H_p. The aim of this paper is to combine the quaternionic Cayley transformation and the properties of H_p on the unit sphere S of $\mathbb{H}$ in order to define directional Hilbert operators H_p^3 on the three–dimensional space $\mathbb{H}_0 = \langle i, j, k\rangle \cong \mathbb{R}^3$.

Let Ω be a smooth domain in $\mathbb{C}^2$. Let $\mathbb{H}$ be the space of real quaternions $q = x_0 + ix_1 + jx_2 + kx_3$, where i, j, k denote the basic quaternions. We identify $\mathbb{H}$ with $\mathbb{C}^2$ by means of the mapping that associates the quaternion $q = z_1 + z_2 j$ with the pair $(z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$. We consider the class $\mathcal{R}(\Omega)$ of *left–regular* (also called *hyperholomorphic*) functions $f : \Omega \to \mathbb{H}$ in the kernel of the Cauchy–Riemann operator

$$\mathcal{D} = 2\left(\frac{\partial}{\partial \bar z_1} + j\frac{\partial}{\partial \bar z_2}\right) = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3}.$$

This differential operator is a variant of the original Cauchy–Fueter operator (cf. for example [22] and [7, 8])

$$\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}.$$

Hyperholomorphic functions have been studied by many authors (see for instance [1, 10, 14, 20, 21]). Regular functions in the space $\mathcal{R}(\Omega)$ have some characteristics that are more intimately related to the theory of holomorphic functions of two complex variables.

This space contains the identity mapping and any holomorphic map (f_1, f_2) on Ω defines a regular function $f = f_1 + f_2 j$. This is no longer true if we adopt the original definition of Fueter regularity. The space $\mathcal{R}(\Omega)$ exhibits other interesting links with the theory of two complex variables. In particular, it contains the spaces of holomorphic maps with respect to any constant complex structure, not only the standard one.

Let J_1, J_2 be the complex structures on the tangent bundle $T\mathbb{H} \simeq \mathbb{H}$ defined by left multiplication by i and j. Let J_1^*, J_2^* be the dual structures on the cotangent bundle $T^*\mathbb{H} \simeq \mathbb{H}$ and set $J_3^* = J_1^* J_2^*$. For every complex structure $J_p = p_1 J_1 + p_2 J_2 + p_3 J_3$ (p a imaginary unit in the unit sphere $\mathbb{S}^2$), let L_p be the complex structure defined by left multiplication by p and

$$\overline{\partial}_p = \frac{1}{2}\left(d + pJ_p^* \circ d\right)$$

the Cauchy–Riemann operator w.r.t. the structures J_p and L_p. Let $\mathrm{Hol}_p(\Omega, \mathbf{H}) = \mathrm{Ker}\,\overline{\partial}_p$ be the space of holomorphic maps from (Ω, J_p) to $(\mathbf{H}, L_p)$. Then every element of $\mathrm{Hol}_p(\Omega, \mathbf{H})$ is regular.

These subspaces do not fill the whole space of regular functions: it was proved in [13] that there exist regular functions that are not holomorphic for any p.

In Section 3 we recall some results about the action of the conformal group of $\mathbb{H}$ on regular functions. We refer to [17] for complete proofs and other applications. Some of the results we describe can be deduced from [22] (Theorem 6) using the reflection $\gamma(z_1, z_2) = (z_1, \bar{z}_2)$. We recall the definition of the *quaternionic Cayley transformation* $\psi(q) = (q+1)(1-q)^{-1}$, which maps diffeomorphically the unit ball B to the right half–space $\mathbb{H}^+ = \{q \in \mathbb{H} \mid Re(q) > 0\}$. We refer to [5] for geometric properties of ψ.

The construction of the directional Hilbert operators makes use of the rotational properties of regular functions (see §3.2), which were firstly studied in [22] in the context of Fueter-regularity. This allows to reduce some definitions to the standard complex structure.

In Section 4 we prove our main result. After having recalled the construction of the directional, p-dependent, Hilbert operator H_p on the unit sphere $S = \partial B$, we define the three–dimensional operator H_p^3 by means of the Cayley transformation.

We introduce a Sobolev–type space $W^1_{\overline{\partial}_p}(\mathbb{H}_0, \mathbb{H})$ of $\mathbb{H}$–valued functions f, of class $L^2(\mathbb{H}_0)$, defined in terms of the Cayley transformation.

In Theorem 9 we prove that for every $p \in \mathbb{S}^2$, there exists a $\mathbb{H}$–linear bounded Hilbert operator H_p^3 on the space $W^1_{\overline{\partial}_p}(\mathbb{H}_0, \mathbb{H})$. For every $f \in W^1_{\overline{\partial}_p}(\mathbb{H}_0, \mathbb{H})$, the function $R_p^3(f) := f + H_p^3(f)$ is the trace of a regular function on $\mathbb{H}^+$. Functions f in the kernel of H_p^3 are in a one–to–one correspondence with CR_p–functions on S.

2. Notations and definitions

2.1. Fueter regular functions

We identify the space $\mathbb{C}^2$ with the set $\mathbb{H}$ of quaternions by means of the mapping that associates the pair $(z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$ with the quaternion $q = z_1 + z_2 j = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H}$. A quaternionic function $f = f_1 + f_2 j \in C^1(\Omega)$ is *(left) regular* (or *hyperholomorphic*) on Ω if

$$\mathcal{D}f = 2\left(\frac{\partial}{\partial \bar{z}_1} + j\frac{\partial}{\partial \bar{z}_2}\right) = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} - k\frac{\partial f}{\partial x_3} = 0 \quad \text{on } \Omega.$$

We will denote by $\mathcal{R}(\Omega)$ the space of regular functions on Ω. The space $\mathcal{R}(\Omega)$ contains the identity mapping and every holomorphic mapping (f_1, f_2) on Ω defines a regular function $f = f_1 + f_2 j$. We recall some properties of regular functions, for which we refer to the papers of Sudbery[22], Shapiro and Vasilevski[20] and Kravchenko and Shapiro[10]:

1. The complex components are both holomorphic or both non–holomorphic.
2. Every regular function is harmonic.
3. If Ω is pseudoconvex, every complex harmonic function is the complex component of a regular function on Ω.

4. The space $\mathcal{R}(\Omega)$ of regular functions on Ω is a *right* $\mathbb{H}$–module with integral representation formulas.
5. f is regular $\Leftrightarrow$ $\quad \dfrac{\partial f_1}{\partial \bar z_1} = \dfrac{\partial \overline{f_2}}{\partial z_2}, \quad \dfrac{\partial f_1}{\partial \bar z_2} = -\dfrac{\partial \overline{f_2}}{\partial z_1}.$
6. A regular function can have rank $0, 2, 3$ or 4 but not rank 1.

Joyce introduced in [9] the module of *q–holomorphic* functions on a hypercomplex manifold. This definition is equivalent to regularity on $\mathbb{H}$. A hypercomplex structure on the manifold $\mathbb{H}$ is given by the complex structures J_1, J_2 on $T\mathbb{H} \simeq \mathbb{H}$ defined by left multiplication by i and j. Let J_1^*, J_2^* be the dual structures on $T^*\mathbb{H} \simeq \mathbb{H}$ and set $J_3^* = J_1^* J_2^*$, which is equivalent to $J_3 = -J_1 J_2$. A function f is regular if and only if f is *q–holomorphic*, i.e.

$$df + iJ_1^*(df) + jJ_2^*(df) + kJ_3^*(df) = 0.$$

In complex components $f = f_1 + f_2 j$, we can rewrite the equations of regularity as

$$\overline{\partial} f_1 = J_2^*(\partial \overline{f}_2).$$

The original definition of regularity given by Fueter (cf. [22] or [7]) differs from the one adopted here by a real coordinate reflection. Let γ be the transformation of $\mathbb{C}^2$ defined by $\gamma(z_1, z_2) = (z_1, \bar z_2)$. Then a C^1 function f is regular on the domain Ω if and only if $f \circ \gamma$ is Fueter–regular on $\gamma(\Omega) = \gamma^{-1}(\Omega)$, i.e. it satisfies the differential equation

$$\left(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}\right)(f \circ \gamma) = 0 \quad \text{on } \gamma^{-1}(\Omega).$$

2.2. Holomorphic functions w.r.t. a complex structure J_p

Let $J_p = p_1 J_1 + p_2 J_2 + p_3 J_3$ be the orthogonal complex structure on $\mathbb{H}$ defined by a unit imaginary quaternion $p = p_1 i + p_2 j + p_3 k$ in the sphere $\mathbb{S}^2 = \{p \in \mathbb{H} \mid p^2 = -1\}$. In particular, J_1 is the standard complex structure of $\mathbb{C}^2 \simeq \mathbb{H}$.

Let $\mathbb{C}_p = \langle 1, p\rangle$ be the complex plane spanned by 1 and p and let L_p be the complex structure defined on $T^*\mathbb{C}_p \simeq \mathbb{C}_p$ by left multiplication by p. We have $L_p = J_{\gamma(p)}$, where $\gamma(p) = p_1 i + p_2 j - p_3 k$.

Let $\mathrm{Hol}_p(\Omega, \mathbb{H})$ be the space of holomorphic maps from (Ω, J_p) to $(\mathbb{H}, L_p)$

$$\mathrm{Hol}_p(\Omega, \mathbb{H}) = \{f : \Omega \to \mathbb{H} \mid \overline{\partial}_p f = 0 \text{ on } \Omega\} = \mathrm{Ker}\, \overline{\partial}_p$$

where $\overline{\partial}_p$ is the Cauchy–Riemann operator

$$\overline{\partial}_p = \frac{1}{2}\left(d + pJ_p^* \circ d\right).$$

These functions will be called *J_p–holomorphic maps* on Ω. For any positive orthonormal basis $\{1, p, q, pq\}$ of $\mathbb{H}$ ($p, q \in \mathbb{S}^2$), let $f = f_1 + f_2 q$ be the decomposition of f with respect to the orthogonal sum

$$\mathbb{H} = \mathbb{C}_p \oplus (\mathbb{C}_p)q.$$

Let $f_1 = f^0 + pf^1$, $f_2 = f^2 + pf^3$, with f^0, f^1, f^2, f^3 the real components of f w.r.t. the basis $\{1, p, q, pq\}$. Then the equations of regularity can be rewritten in complex form as

$$\overline{\partial}_p f_1 = J_q^*(\partial_p \overline{f}_2),$$

where $\overline{f}_2 = f^2 - pf^3$ and $\partial_p = \frac{1}{2}\left(d - pJ_p^* \circ d\right)$. Therefore every $f \in \mathrm{Hol}_p(\Omega, \mathbb{H})$ is a regular function on Ω.

Remark 1.
1. The identity map belongs to the space $\mathrm{Hol}_i(\Omega, \mathbb{H}) \cap \mathrm{Hol}_j(\Omega, \mathbb{H})$ but not to $\mathrm{Hol}_k(\Omega, \mathbb{H})$.
2. For every $p \in \mathbb{S}^2$, $\mathrm{Hol}_{-p}(\Omega, \mathbb{H}) = \mathrm{Hol}_p(\Omega, \mathbb{H})$.
3. Every $\mathbb{C}_p$–valued regular function is a J_p–holomorphic function.
4. If $f \in \mathrm{Hol}_p(\Omega, \mathbb{H}) \cap \mathrm{Hol}_q(\Omega, \mathbb{H})$, with $p \neq \pm q$, then $f \in \mathrm{Hol}_r(\Omega, \mathbb{H})$ for every $r = \frac{\alpha p + \beta q}{\|\alpha p + \beta q\|}$ $(\alpha, \beta \in \mathbb{R})$ in the circle of $\mathbb{S}^2$ generated by p and q.

In [13] was proved that on every domain Ω there exist regular functions that are not J_p-holomorphic for any p. The criterion for holomorphicity is based on an energy–minimizing property of holomorphic maps. The *energy quadric* of a regular function f (cf. [15]) is a positive semi–definite quadric, defined by means of the Lichnerowicz homotopy invariants, which contains information about the holomorphicity properties of the function.

Examples 1.
1. *$f = \bar{z}_1 + z_2 + \bar{z}_2 j$ is J_p-holomorphic, with $p = \frac{1}{\sqrt{5}}(i - 2k)$.*
2. *$f = z_1 + z_2 + \bar{z}_1 + (z_1 + z_2 + \bar{z}_2)j$ is regular, but not holomorphic.*
3. *(Nonlinear case) $f = |z_1|^2 - |z_2|^2 + \bar{z}_1\bar{z}_2 j$ is regular but not holomorphic w.r.t. any complex structure J_p.*

2.3. Cauchy–Riemann operators

Let $\Omega = \{z \in \mathbb{C}^2 : \rho(z) < 0\}$ be a domain with C^∞–smooth boundary in $\mathbb{C}^2$. We assume ρ of class C^∞ on $\mathbb{C}^2$ and $d\rho \neq 0$ on $\partial\Omega$. For every complex valued function $g \in C^1(\overline{\Omega})$, we can define on a neighborhood of $\partial\Omega$ the normal components of ∂g and $\overline{\partial} g$

$$\partial_n g = \sum_k \frac{\partial g}{\partial z_k} \frac{\partial \rho}{\partial \bar{z}_k} \frac{1}{|\partial \rho|} \quad \text{and} \quad \overline{\partial}_n g = \sum_k \frac{\partial g}{\partial \bar{z}_k} \frac{\partial \rho}{\partial z_k} \frac{1}{|\partial \rho|},$$

where $|\partial\rho|^2 = \sum_{k=1}^2 \left|\frac{\partial \rho}{\partial z_k}\right|^2$. By means of the Hodge $*$–operator and the Lebesgue surface measure $d\sigma$, we can also write

$$\overline{\partial}_n g \, d\sigma = * \overline{\partial} g_{|\partial\Omega}.$$

In a neighbourhood of $\partial\Omega$ we have the decomposition of $\overline{\partial} g$ in the tangential and the normal parts

$$\overline{\partial} g = \overline{\partial}_t g + \overline{\partial}_n g \frac{\overline{\partial}\rho}{|\overline{\partial}\rho|}.$$

Let $\mathcal{L}$ be the tangential Cauchy–Riemann operator

$$\mathcal{L} = \frac{1}{|\partial\rho|}\left(\frac{\partial\rho}{\partial\bar{z}_2}\frac{\partial}{\partial\bar{z}_1} - \frac{\partial\rho}{\partial\bar{z}_1}\frac{\partial}{\partial\bar{z}_2}\right).$$

The tangential part of $\overline{\partial} g$ is related to $\mathcal{L} g$ by the following formula

$$\overline{\partial}_t g \wedge d\zeta_{|\partial\Omega} = 2\mathcal{L} g \, d\sigma.$$

A complex function $g \in C^1(\partial\Omega)$ is a *CR–function* if and only if $\mathcal{L}g = 0$ on $\partial\Omega$. Notice that $\overline{\partial} g$ has coefficients of class $L^2(\partial\Omega)$ if and only if both $\overline{\partial}_n g$ and $\mathcal{L}g$ are of class $L^2(\partial\Omega)$.

If $g = g_1 + g_2 j$ is a regular function of class C^1 on Ω, then the equations $\overline{\partial}_n g_1 = -\overline{\mathcal{L}(g_2)}$, $\overline{\partial}_n g_2 = \overline{\mathcal{L}(g_1)}$ hold on $\partial\Omega$. Conversely, a harmonic function f of class $C^1(\Omega)$ is regular if it satisfies these equations on $\partial\Omega$ (cf. [14]). If Ω has connected boundary, it is sufficient that one of the equations is satisfied.

In place of the standard complex structure J_1, we can take on $\mathbb{C}^2$ a different complex structure J_p and consider the corresponding Cauchy–Riemann operators. We will denote by $\partial_{p,n}$ and $\overline{\partial}_{p,n}$ the normal components of ∂_p and $\overline{\partial}_p$ respectively, by $\overline{\partial}_{p,t}$ the tangential component of $\overline{\partial}_p$ and by $\mathcal{L}_p$ the tangential Cauchy–Riemann operator with respect to the structure J_p. Then we have the relations

$$\begin{aligned}\overline{\partial}_p g &= \overline{\partial}_{p,t}\, g + \overline{\partial}_{p,n} g \frac{\overline{\partial}_p \rho}{|\overline{\partial}_p \rho|},\\ \overline{\partial}_{p,t}\, g \wedge d\zeta_{|\partial\Omega} &= 2\mathcal{L}_p g\, d\sigma,\\ \overline{\partial}_{p,n} g\, d\sigma &= *\,\overline{\partial}_p g_{|\partial\Omega}.\end{aligned}$$

The space

$$CR_p(\partial\Omega) = \operatorname{Ker}\mathcal{L}_p = \{g : \partial\Omega \to \mathbb{C}_p \mid \mathcal{L}_p g = 0\}$$

has elements the CR–functions on $\partial\Omega$ with respect to the operator $\overline{\partial}_p$.

Remark 2. The operators $\overline{\partial}_p$, $\partial_{p,n}$, $\overline{\partial}_{p,n}$ and $\mathcal{L}_p$ are $\mathbb{C}_p$–linear and they map $\mathbb{C}_p$–valued functions of class C^1 to continuous $\mathbb{C}_p$–valued functions.

3. Regular functions and conformal mappings

In this section we recall some results about the action of the conformal group of $\mathbb{H}$ on regular functions. We refer to [17] for complete proofs and more applications. Some of the results we describe can be deduced from [22] (Theorem 6) using the reflection $\gamma(z_1, z_2) = (z_1, \bar{z}_2)$ introduced in §2.1.

We recall some definitions and properties of conformal and orientation preserving mappings of the one–point compactification $\mathbb{H}^*$ of $\mathbb{H}$, for which we refer to [5], [7]§6.2, [19] and [22] and to the references cited in those papers. The *Dieudonné determinant* of a quaternionic matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the real non–negative number

$$\det_{\mathbb{H}}(A) = \sqrt{|a|^2|d|^2 + |b|^2|c|^2 - 2Re(c\bar{a}b\bar{d})}.$$

It satisfies Binet property $\det_{\mathbb{H}}(AB) = \det_{\mathbb{H}}(A)\det_{\mathbb{H}}(B)$ and a matrix A is (left and right) invertible if and only if $\det_{\mathbb{H}} A \neq 0$. Then we can consider the general linear group

$$GL(2, \mathbb{H}) = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ quaternionic matrix of order 2} \mid \det_{\mathbb{H}} A \neq 0 \right\}.$$

From a theorem of Liouville, the general conformal transformation of $\mathbb{H}^*$ is a quaternionic *Möbius transformation*, i.e. a fractional linear map of the form

$$L_A(q) = (aq+b)(cq+d)^{-1}, \quad A \in GL(2,\mathbb{H}).$$

The matrix A is determined by L_A up to a real scalar multiple. For every pair of matrices $A, B \in GL(2,\mathbb{H})$, $L_A \circ L_B = L_{AB}$. We have also the alternative representation of conformal mappings

$$L'_A(q) = (qc+d)^{-1}(qa+b), \quad \det_{\mathbb{H}} \bar{A} \neq 0.$$

Proposition 1. *Given $f \in C^1(\Omega)$ and a conformal transformation $L_A(q) = (aq+b)(cq+d)^{-1}$, let f^A be the function*

$$f^A(q) = \frac{(c\gamma(q)+d)^{-1}}{|c\gamma(q)+d|^2} f(L'_{\gamma(A)}(q)),$$

where $\gamma(A) = \begin{bmatrix} \gamma(a) & \gamma(b) \\ \gamma(c) & \gamma(d) \end{bmatrix}$. Then f is regular on Ω if and only if f^A is regular on $\Omega' = (L'_{\gamma(A)})^{-1}(\Omega)$. Moreover, $(f^A)^B = f^{AB}$ for every $A, B \in GL(2,\mathbb{H})$.

Proof. The first statement can be deduced from the result of Sudbery (cf. [22] Theorem 6), since $f \in \mathcal{R}(\Omega)$ iff $F = f \circ \gamma$ is Fueter–regular on $\gamma(\Omega)$. This last condition is equivalent to the Fueter–regularity of the transformed function

$$F^A(p) = \frac{(cp+d)^{-1}}{|cp+d|^2} F(L_A(p))$$

on $(L_A)^{-1}(\gamma(\Omega))$. Note that this function differs from the one given by Sudbery by a real constant factor. We then obtain that f is regular iff $F^A \circ \gamma$ is regular. We have

$$F^A \circ \gamma(q) = \frac{(c\gamma(q)+d)^{-1}}{|c\gamma(q)+d|^2} f \circ \gamma \circ L_A \circ \gamma(q) = f^A(q),$$

since $\gamma \circ L_A \circ \gamma(q) = L'_{\gamma(A)}(q)$. The last statement of the theorem is a straightforward computation using the equality

$$L'_{\gamma(A)} \circ L'_{\gamma(B)} = (\gamma \circ L_A \circ \gamma) \circ (\gamma \circ L_B \circ \gamma) = \gamma \circ L_{AB} \circ \gamma = L'_{\gamma(AB)}.$$

□

Remark 3. We can restrict the choice of the matrix A to the subgroup $SL(2,\mathbb{H}) = \{A \in GL(2,\mathbb{H}) \mid \det_{\mathbb{H}}(A) = 1\}$. In this case, the same conformal transformation gives rise to two functions, f^A and $f^{-A} = -f^A$.

Example 1. *Given two unit quaternions $a, d \in \mathbb{H}$, the diagonal matrix $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ induces the* four–dimensional rotation $q \mapsto aqd^{-1}$. *Given a regular function f on Ω, the function*

$$f^A(q) = d^{-1} f(\gamma(d)^{-1} q \gamma(a))$$

is regular on $\Omega' = \gamma(d)\Omega\gamma(a)^{-1}$.

3.1. The Cayley transformation

The quaternionic Cayley transformation $\psi(q) = (q+1)(1-q)^{-1}$ maps diffeomorphically the unit ball B to the right half–space $\mathbb{H}^+ = \{q \in \mathbb{H} \mid Re(q) > 0\}$ (see [5] for geometric properties of ψ). It transforms regular functions f on $\mathbb{H}^+$ into

$$f^\psi(q) = 2^{3/2} \frac{(1-\gamma(q))^{-1}}{|1-\gamma(q)|^2} f(\psi(q)),$$

regular on B. Here ψ corresponds to the matrix $C = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \in SL(2, \mathbb{H})$. The inverse mapping $\phi(q) = (q-1)(1+q)^{-1}$ transforms $g \in \mathcal{R}(B)$ into

$$g^\phi(q) = 2^{3/2} \frac{(1+\gamma(q))^{-1}}{|1+\gamma(q)|^2} g(\phi(q)) \in \mathcal{R}(\mathbb{H}^+).$$

The factor $2^{3/2}$ in the formulas has been chosen to get $(f^\psi)^\phi = f$. The maps $f \mapsto f^\psi$ and $g \mapsto g^\phi$ are right $\mathbb{H}$–linear.

The extension of ϕ to the boundary $\mathbb{H}_0 = \partial \mathbb{H}^+$ maps diffeomorphically $\mathbb{H}_0$ onto $S \setminus \{1\}$. We will denote again by ϕ and ψ these extensions.

3.2. Rotated regular functions

A unit quaternion d defines the *three–dimensional rotation* $q \mapsto rot_d(q) := dqd^{-1}$, which gives rise to the function (cf. Example 1)

$$f^A(q) = d^{-1} f(\gamma(d)^{-1} q \gamma(d)),$$

where A is the scalar matrix $A = \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix}$. Taking $d = \gamma(a)^{-1}$ and multiplying by $\gamma(a)^{-1}$ on the right, we obtain the function $f^a = rot_{\gamma(a)} \circ f \circ rot_a$.

Proposition 2 ([16, 17])**.** *Let $f \in C^1(\Omega)$ and let $a \in \mathbb{H}$, $a \neq 0$. Let $rot_a(q) = aqa^{-1}$ be the three–dimensional rotation of $\mathbb{H}$ defined by a. Let $f^a = rot_{\gamma(a)} \circ f \circ rot_a$. Then*

1. *f is regular on Ω if and only if f^a is regular on $\Omega^a = rot_a^{-1}(\Omega) = a^{-1}\Omega a$.*
2. *f^a is J_p–holomorphic if and only if f is $J_{p'}$–holomorphic, with $p' = rot_{\gamma(a)}^{-1}(p)$.*

Remark 4. The rotated function f^a has the following properties:

1. $(f^a)^b = f^{ab}$ and $(f+g)^a = f^a + g^a$.
2. $(f^a)^{a^{-1}} = f$.
3. $f^{-a} = f^a$.
4. If $b \in \mathbb{H}$, then $(fb)^a = f^a \, rot_{\gamma(a)}(b)$.

Proposition 3. *The action of the Cayley transformation commutes with that of rotations:*

$$(f^\psi)^a = (f^a)^\psi \quad \textit{and} \quad (g^\phi)^a = (g^a)^\phi \ \forall f \in \mathcal{R}(\mathbb{H}^+), g \in \mathcal{R}(B).$$

Proof. $f^a = f^A \gamma(a)^{-1}$, with A a scalar matrix. Since A commutes with the real matrices C and C^{-1}, it follows that

$$(f^a)^\psi = (f^A \gamma(a)^{-1})^\psi = (f^A)^\psi \gamma(a)^{-1} = f^{AC} \gamma(a)^{-1} = (f^C)^A \gamma(a)^{-1} = (f^\psi)^a$$

and similarly for ϕ. □

Rotations also allow to express the relation between the Cauchy–Riemann operators $\overline{\partial}$ and $\overline{\partial}_p$ (cf. §2.3).

Proposition 4 ([16]). *Let* $a \in \mathbb{H}$, $a \neq 0$. *If* $p = \gamma(r_a(i))$ *and* $g : \overline{\Omega} \to \mathbb{C}_p$ *is of class* $C^1(\overline{\Omega})$, *then* $\overline{\partial} g^a = (\overline{\partial}_p g)^a$. *Moreover* $\overline{\partial}_n g^a = (\overline{\partial}_{p,n} g)^a$ *and* $\mathcal{L} g^a = (\mathcal{L}_p g)^a$ *on* $\partial\Omega^a$. *In particular,* $g \in CR_p(\partial\Omega)$ *if and only if* $g^a \in CR(\partial\Omega^a)$.

Remark 5. For a general conformal transformation L_A, the (Dirichlet) energy and, *a fortiori*, the energy quadric of a regular function is not conserved. The same happens for J_p–holomorphicity. In particular, the holomorphicity of g on B does not imply the holomorphicity of g^ϕ on $\mathbb{H}^+$, and conversely.

4. Directional Hilbert operators

For a bounded domain Ω with C^∞–smooth boundary, we consider the following Sobolev–type Hilbert subspace of $L^2(\partial\Omega, \mathbb{C}_p)$:

$$\begin{aligned} W^1_{\overline{\partial}_p}(\partial\Omega, \mathbb{C}_p) &= \{f \in L^2(\partial\Omega, \mathbb{C}_p) \mid \overline{\partial}_p f \in L^2(\partial\Omega, \mathbb{C}_p)\} \\ &= \{f \in L^2(\partial\Omega, \mathbb{C}_p) \mid \overline{\partial}_{p,n} f \text{ and } \mathcal{L}_p f \in L^2(\partial\Omega, \mathbb{C}_p)\} \end{aligned}$$

with product

$$(f,g)_{W^1_{\overline{\partial}_p}} = (f,g) + (\overline{\partial}_{p,n} f, \overline{\partial}_{p,n} g) + (\mathcal{L}_p f, \mathcal{L}_p g),$$

where (f,g) is the $L^2(\partial\Omega)$–product. Here and in the following we always identify $f \in L^2(\partial\Omega)$ with its harmonic extension on Ω. These spaces are vector spaces over $\mathbb{R}$ and over $\mathbb{C}_p$. For every $\alpha > 0$, the space $W^1_{\overline{\partial}_p}(\partial\Omega, \mathbb{C}_p)$ contains, in particular, every $\mathbb{C}_p$–valued function f of class $C^{1+\alpha}(\partial\Omega)$. Indeed, under this regularity condition f has an harmonic extension of class (at least) C^1 on $\overline{\Omega}$.

Let $L^2(\partial\Omega, \mathbb{C}_p^\perp)$ be the space of functions fq, $f \in L^2(\partial\Omega, \mathbb{C}_p)$, where $q \in \mathbb{S}^2$ is any unit orthogonal to p and let

$$W^1_{\overline{\partial}_p}(\partial\Omega, \mathbb{C}_p^\perp) = \{f \in L^2(\partial\Omega, \mathbb{C}_p^\perp) \mid \overline{\partial}_p f \in L^2(\partial\Omega, \mathbb{C}_p^\perp)\}.$$

Then $W^1_{\overline{\partial}_p}(\partial\Omega, \mathbb{C}_p^\perp) = \{fq \mid f \in W^1_{\overline{\partial}_p}(\partial\Omega)\}$ for any $q \in \mathbb{S}^2$ orthogonal to p. On these spaces we consider the products w.r.t. which the right multiplication by q is an isometry:

$$\begin{aligned} (f,g)_{L^2(\partial\Omega, \mathbb{C}_p^\perp)} &= (fq, gq)_{L^2(\partial\Omega, \mathbb{C}_p)}, \\ (f,g)_{W^1_{\overline{\partial}_p}(\partial\Omega, \mathbb{C}_p^\perp)} &= (fq, gq)_{W^1_{\overline{\partial}_p}(\partial\Omega, \mathbb{C}_p)}. \end{aligned}$$

Proposition 5. *The above products are independent of* $q \perp p$.

Proof. Let $q' = aq + bpq \in \mathbb{C}_p^\perp$ be another element of $\mathbb{S}^2$ orthogonal to p, with $a, b \in \mathbb{R}$, $a^2 + b^2 = 1$. If $fq = f^0 + f^1 p$, then $fq' = (af^0 + bf^1) + (af^1 - bf^0)p$. Similarly, $gq' = (ag^0 + bg^1) + (ag^1 - bg^0)p$, from which we get

$$\begin{aligned} (fq', gq')_{L^2(\partial\Omega, \mathbb{C}_p)} &= (af^0 + bf^1, ag^0 + bg^1)_{L^2} + (af^1 - bf^0, ag^1 - bg^0)_{L^2} \\ &= (a^2 + b^2)(f^0, g^0)_{L^2} + (a^2 + b^2)(f^1, g^1)_{L^2} = (fq, gq)_{L^2(\partial\Omega, \mathbb{C}_p)}. \end{aligned}$$

The independence of the second product follows from that of the first. □

We will consider also the space of $\mathbb{H}$–valued functions

$$W^1_{\overline{\partial}_p}(\partial\Omega,\mathbb{H}) = \{f \in L^2(\partial\Omega,\mathbb{H}) \mid \overline{\partial}_p f \in L^2(\partial\Omega,\mathbb{H})\}$$

with norm

$$\|f\|_{W^1_{\overline{\partial}_p}(\partial\Omega,\mathbb{H})} = \left(\|f_1\|^2_{W^1_{\overline{\partial}_p}(\partial\Omega,\mathbb{C}_p)} + \|f_2\|^2_{W^1_{\overline{\partial}_p}(\partial\Omega,\mathbb{C}_p)}\right)^{1/2},$$

where $f = f_1 + f_2 q \in W^1_{\overline{\partial}_p}(\partial\Omega,\mathbb{C}_p) \oplus W^1_{\overline{\partial}_p}(\partial\Omega,\mathbb{C}_p^{\perp})$, $f_i \in W^1_{\overline{\partial}_p}(\partial\Omega,\mathbb{C}_p)$ and q is any imaginary unit orthogonal to p. It follows from Proposition 5 that this norm does not depends on q.

We recall the definition of directional Hilbert operators introduced in [16]. For every $\mathbb{C}_p$–valued function f_1 in $W^1_{\overline{\partial}_p}(\partial\Omega,\mathbb{C}_p)$ and every fixed $q \in \mathbb{S}^2$ orthogonal to p, there exists a function $H_{p,q}(f_1) : \partial\Omega \to \mathbb{C}_p$ in the same space as f_1, such that $f = f_1 + H_{p,q}(f_1)q$ is the boundary value of a regular function on Ω. f_1 and $H_{p,q}(f_1)$ are called *quaternionic harmonic conjugates*. The function $H_{p,q}(f_1)$ is uniquely characterized by $L^2(\partial\Omega)$–orthogonality to the space of CR–functions with respect to the structure J_p. Moreover, $H_{p,q}$ is a bounded operator on the space $W^1_{\overline{\partial}_p}(\partial\Omega,\mathbb{C}_p)$.

For every fixed direction p, it is also possible to choose a quaternionic regular harmonic conjugate of f_1 in a way independent of the chosen orthogonal direction q. Taking restrictions to the boundary $\partial\Omega$, this construction permits to define the directional, p–dependent, Hilbert operator H_p.

4.1. The case of the unit sphere

In this section we recall the more precise results which can be obtained on the unit sphere S.

Theorem 6 ([16]§7). *Given a $\mathbb{C}_p$–valued function $f_1 \in W^1_{\overline{\partial}_p}(S,\mathbb{C}_p)$, there exists $H_p(f_1) \in W^1_{\overline{\partial}_p}(S,\mathbb{C}_p^{\perp})$ such that $f = f_1 + H_p(f_1)$ is the trace of a regular function on B. Moreover, $H_p(f_1)$ satisfies the estimate*

$$\|H_p(f_1)\|_{W^1_{\overline{\partial}_p}(S,\mathbb{C}_p^{\perp})} \le \left(2\|\overline{\partial}_{p,n} f_1\|^2_{L^2(S)} + \|\mathcal{L}_p f_1\|^2_{L^2(S)}\right)^{1/2}.$$

The operator $H_p : W^1_{\overline{\partial}_p}(S,\mathbb{C}_p) \to W^1_{\overline{\partial}_p}(S,\mathbb{C}_p^{\perp})$ is a right $\mathbb{C}_p$–linear bounded operator, with kernel $CR_p(S)$.

The operator H_p can be extended by right $\mathbb{H}$–linearity to the space $W^1_{\overline{\partial}_p}(S,\mathbb{H})$. If $f \in W^1_{\overline{\partial}_p}(S,\mathbb{H})$ and q is any imaginary unit orthogonal to p, let $f = f_1 + f_2 q \in W^1_{\overline{\partial}_p}(S,\mathbb{C}_p) \oplus W^1_{\overline{\partial}_p}(S,\mathbb{C}_p^{\perp})$, $f_i \in W^1_{\overline{\partial}_p}(S,\mathbb{C}_p)$. We set

$$H_p(f) = H_p(f_1) + H_p(f_2)q.$$

This definition is independent of q, because if $f = f_1 + f_2' q'$, then $(f_2 q - f_2' q')q$ is a CR_p–function and therefore $0 = H_p(-f_2 - f_2' q' q) = -H_p(f_2) - H_p(f_2')q'q \Rightarrow H_p(f_2)q = H_p(f_2')q'$. The operator H_p will be called a *directional Hilbert operator* on S.

Corollary 7. *The Hilbert operator $H_p : W^1_{\overline{\partial}_p}(S, \mathbb{H}) \to W^1_{\overline{\partial}_p}(S, \mathbb{H})$ is right $\mathbb{C}_p$–linear and $\mathbb{H}$–linear, its kernel is the space of $\mathbb{H}$–valued CR_p–functions and satisfies the estimate*

$$\|H_p(f)\|_{W^1_{\overline{\partial}_p}(S,\mathbb{H})} \leq \sqrt{2}\, \|f\|_{W^1_{\overline{\partial}_p}(S,\mathbb{H})}.$$

For every $f \in W^1_{\overline{\partial}_p}(S, \mathbb{H})$, the function $R_p(f) := f + H_p(f)$ is the trace of a regular function on B.

4.2. The case of the three–dimensional space $\mathbb{H}_0$

Now we come to our main result. We introduce the following function spaces on the three–dimensional space $\mathbb{H}_0 = \langle i, j, k\rangle$:

$$W^1_{\overline{\partial}_p}(\mathbb{H}_0, \mathbb{H}) = W^1_{\overline{\partial}_p}(S, \mathbb{H})^\phi := \{f = g^\phi \mid g \in W^1_{\overline{\partial}_p}(S, \mathbb{H})\}$$

with product

$$(f, f')_{W^1_{\overline{\partial}_p}(\mathbb{H}_0,\mathbb{H})} = (f^\psi, f'^\psi)_{W^1_{\overline{\partial}_p}(S,\mathbb{H})}.$$

Proposition 8. *A function g belongs to the space $L^2(S, \mathbb{H})$ if and only if $f = g^\phi$ belongs to $L^2(\mathbb{H}_0, \mathbb{H})$. Therefore $W^1_{\overline{\partial}_p}(\mathbb{H}_0, \mathbb{H}) \subseteq L^2(\mathbb{H}_0, \mathbb{H})$ and $f \in W^1_{\overline{\partial}_p}(\mathbb{H}_0, \mathbb{H})$ if and only if $f \in L^2(\mathbb{H}_0, \mathbb{H})$ and $(\overline{\partial}_p(f^\psi))^\phi \in L^2(\mathbb{H}_0, \mathbb{H})$.*

Proof. Let $g = f^\psi$. Then

$$\int_S |g(q)|^2 d\sigma = 8 \int_S \frac{|f(\psi(q))|^2}{|1 - \gamma(q)|^6} d\sigma = 16 \int_{\phi(\mathbb{H}_0)} |f(q')|^2 |1 + q'|^6 d\sigma(\phi(q'))$$

since $|1 - \gamma(q)| = |1 - \phi(q')| = 2|1 + q'|^{-1}$, with $q' = \psi(q) = x_1 i + x_2 j + x_3 k \in \mathbb{H}_0$, $q = \phi(q') \in S$. The Jacobian determinant of $\phi_{|\mathbb{H}_0}$ has order $|q'|^{-6}$ for large $|q'|$. Then

$$\int_S |g(q)|^2 d\sigma \approx \int_{\mathbb{H}_0} \left| \frac{q'}{1 + q'} f(q') \right|^2 dx_1 dx_2 dx_3 = \int_{\mathbb{H}_0} \frac{|q'|^2}{1 + |q'|^2} |f(q')|^2 dx_1 dx_2 dx_3.$$

□

Remark 6. In general, $(\overline{\partial}_p(f^\psi))^\phi \neq \overline{\partial}_p f$. Let $p = \gamma(r_a(i))$. From Propositions 3 and 4 it follows that

$$(\overline{\partial}_p(f^\psi))^a = \overline{\partial}((f^\psi)^a) = \overline{\partial}((f^a)^\psi).$$

Then $W^1_{\overline{\partial}_p}(\mathbb{H}_0, \mathbb{H})^a := \{f^a \mid f \in W^1_{\overline{\partial}_p}(\mathbb{H}_0, \mathbb{H})\} = W^1_{\overline{\partial}}(\mathbb{H}_0, \mathbb{H})$.

The space $W^1_{\overline{\partial}_p}(\mathbb{H}_0, \mathbb{H})$ contains the subspace of the rational functions g^ϕ, g a polynomial function (see the examples in this section).

Theorem 9. *For every $p \in \mathbb{S}^2$, there exists a Hilbert operator $H^3_p : W^1_{\overline{\partial}_p}(\mathbb{H}_0, \mathbb{H}) \to W^1_{\overline{\partial}_p}(\mathbb{H}_0, \mathbb{H})$ which is right $\mathbb{C}_p$–linear and $\mathbb{H}$–linear and satisfies the estimate*

$$\|H^3_p(f)\|_{W^1_{\overline{\partial}_p}(\mathbb{H}_0,\mathbb{H})} \leq \sqrt{2}\,\|f\|_{W^1_{\overline{\partial}_p}(\mathbb{H}_0,\mathbb{H})}.$$

For every $f \in W^1_{\overline{\partial}_p}(\mathbb{H}_0, \mathbb{H})$, the function $R^3_p(f) := f + H^3_p(f)$ is the trace of a regular function on $\mathbb{H}^+$. A function $f \in W^1_{\overline{\partial}_p}(\mathbb{H}_0, \mathbb{H})$ is in the kernel of H^3_p if and only if f^ψ is a CR_p–function on S.

Proof. Given $f \in W^1_{\overline{\partial}_p}(\mathbb{H}_0, \mathbb{H})$, we apply Corollary 7 and set $H^3_p(f) := (H_p(f^\psi))^\phi \in W^1_{\overline{\partial}_p}(\mathbb{H}_0, \mathbb{H})$. By definition, the correspondence $g \longleftrightarrow g^\phi$ is an isometry between the spaces $W^1_{\overline{\partial}_p}(S, \mathbb{H})$ and $W^1_{\overline{\partial}_p}(\mathbb{H}_0, \mathbb{H})$, and the estimate follows. The function $R_p(f^\psi) = f^\psi + H_p(f^\psi)$ is the trace on S of a regular function on B. Then $R^3_p(f) = f + H^3_p(f) = (R_p(f^\psi))^\phi$ is the trace on $\mathbb{H}_0$ of a regular function on $\mathbb{H}^+$. ☐

The Hilbert operator H^3_p on $\mathbb{H}_0$ can be expressed in terms of H^3_i (i.e. by means of the standard complex structure) using rotations. Let $p = \gamma(r_a(i))$. It can be shown that $H_p(g)^a = H_i(g^a)$ for every $g \in W^1_{\overline{\partial}_p}(S, \mathbb{H})$. Therefore

$$(H^3_p(f)^a)^\psi = (H^3_p(f)^\psi)^a = (H_p(f^\psi))^a = H_i((f^\psi)^a) = H_i((f^a)^\psi) = H^3_i(f^a)^\psi$$

for every $f \in W^1_{\overline{\partial}_p}(\mathbb{H}_0, \mathbb{H})$. Then $H^3_p(f)^a = H^3_i(f^a)$.

Examples 2. 1. *Let $f = c^\phi$ be the Cayley transform of a constant quaternionic function ($c \in \mathbb{H}$). We have*

$$\mathbb{H} = \cap_{p\in\mathbb{S}^2} CR_p(S), \quad \mathbb{H}^\phi = \langle 1^\phi \rangle = \{1^\phi c \mid c \in \mathbb{H}\} = \cap_{p\in\mathbb{S}^2} CR_p(S)^\phi.$$

Then $H^3_p(c^\phi) = 0$ for every direction $p \in \mathbb{S}^2$, since $H_p(c) = 0$ on S. The function 1^ϕ, regular on $\mathbb{H}^+$, has trace on $\mathbb{H}_0 = \mathbb{R}^3$ given by

$$1^\phi_{|\mathbb{H}_0}(x_1, x_2, x_3) = \frac{2\sqrt{2}}{(1 + |x|^2)^2}(1, -x_1, -x_2, x_3)$$

and has $L^2(\mathbb{R}^3)$ squared norm equal to $2\pi^2 = Vol(S) = \|1\|^2_{L^2(S)}$.

2. *Let $f = z^\phi_1$. Then*

$$f_{|\mathbb{H}_0}(x_1, x_2, x_3) = \frac{2\sqrt{2}}{(1 + |x|^2)^3}(-1 + 3x_1^2 + x_2^2 + x_3^2,\ 3 - x_1^2 - x_2^2 - x_3^2,$$
$$x_2 + 2x_1x_3 - x_2^3 - x_1^2x_2 - x_3^2x_2,\ -1 + 2x_1x_2 + x_1^2x_2 + x_2^2x_3 + x_3^3)$$

has Hilbert transforms $H^3_i(f) = 0$, since $z_1 \in \mathrm{Hol}_i$, while

$$H^3_j(f) = \frac{\sqrt{2}}{(1 + |x|^2)^3}(-1 + 3x_1^2 - x_2^2 + 3x_3^2,\ 3x_1 + 4x_2x_3 - x_1^3 - x_1x_2^2 - x_1x_3^2,$$
$$- x_2(1 + |x|^2),\ -3x_3 + 4x_1x_2 + x_3^3 + x_3x_1^2 + x_3x_2^2)$$

3. *Let* $f = \bar{z}_1^{\phi}$*, which is* not *regular on* $\mathbb{H}^+$*. Then the Hilbert transform* $H_i^3(f)$ *gives the function*

$$
\begin{aligned}
R_i^3(f) = \frac{2\sqrt{2}}{(1+|x|^2)^3}&(-1 - x_1^2 + 3x_2^2 + 3x_3^2,\ -x_1(1 + x_1^2 + x_2^2 + x_3^2),\\
&3x_2 - 4x_1x_3 - x_2^3 - x_2x_1^2 - x_2x_3^2,\ -3x_3 - 4x_1x_2 + x_1^2x_2 + x_2^2x_3 + x_3^3)
\end{aligned}
$$

which is the trace on $\mathbb{H}_0$ *of the regular function*

$$
\begin{aligned}
F(z_1, z_2) = \frac{2\sqrt{2}}{(|1+z_1|^2 + |z_2|^2)^3}&(-1 - z_1 + \bar{z}_1^2 + z_1\bar{z}_1^2 + (3+\bar{z}_1)|z_2|^2\\
&+ (\bar{z}_2(3 - z_1 + 3\bar{z}_1 - |z_1|^2 - |z_2|^2)j
\end{aligned}
$$

defined on an open set containing the closure of the right–half space $\mathbb{H}^+$.

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