An application of biregularity to quaternionic Lagrange interpolation

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Fueter-regular functions

- $\mathbb{H} \simeq \mathbb{C}^2$: $\quad \mathbb{C}^2 \ni z = (z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$

$$\longleftrightarrow q = z_1 + z_2 j = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H}$$

- Ω bounded domain in $\mathbb{H}$. A quaternionic function $f = f_1 + f_2 j \in C^1(\Omega)$ is (left) regular on Ω if it is in the kernel of the Cauchy-Riemann-Fueter operator

$$\mathcal{D} = 2\left(\frac{\partial}{\partial \bar{z}_1} + j\frac{\partial}{\partial \bar{z}_2}\right) = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3} \quad \text{on } \Omega$$

- Every (standard) holomorphic map $(f_1, f_2) : \Omega \to \mathbb{C}^2$ defines a regular function $f = f_1 + f_2 j$
- Every holomorphic map w.r.t. the structure defined by *left multiplication by j* defines a regular function $(f_1', f_2') : \Omega \to \mathbb{C}_j^2$ holomorphic $\Rightarrow f = f_1' + f_2' i$ regular

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- The space $\mathcal{R}(\Omega)$ of regular functions on Ω is the smallest right $\mathbb{H}$–module defined by a 1^{st}-order differential operator containing the (standard) holomorphic maps and also the holomorphic maps w.r.t. the complex structure given by left multiplication by j

Hypercomplex structure on $\mathbb{H}$

- Hypercomplex structure on $\mathbb{H} \simeq \mathbb{C}^2$: J_1, J_2 complex structures on $T\mathbb{H} \simeq \mathbb{H}$ defined by left multiplication by i and j
  J_1^*, J_2^* dual structures on $T^*\mathbb{H}$. We make the choice
  $J_3^* = J_1^* J_2^* \Rightarrow J_3 = -J_1 J_2$
- We can rewrite the equations of regularity (Joyce 1998)

$$
\begin{aligned}
& df + iJ_1^*(df) + jJ_2^*(df) + kJ_3^*(df) = 0 \\
\text{or} \quad & df^0 = J_1^*(df^1) + J_2^*(df^2) + J_3^*(df^3)
\end{aligned}
$$

or, in complex components $f = f_1 + f_2 j$,

$$\overline{\partial} f_1 = J_2^*(\partial \overline{f}_2)$$

Holomorphic maps w.r.t. a complex structure J_p

Let $J_p = p_1 J_1 + p_2 J_2 + p_3 J_3$ be the orthogonal complex structure on $\mathbb{H}$ defined by $p = p_1 i + p_2 j + p_3 k \in \mathbb{S}^2 = \{p \in \mathbb{H} \mid p^2 = -1\}$.
Let $\mathbb{C}_p = \langle 1, p \rangle$ and L_p the complex structure defined by left multiplication by p. We have $L_p = J_{\gamma(p)}$, where $\gamma(p) = p_1 i + p_2 j - p_3 k$.
Consider J_p-holomorphic maps from (Ω, J_p) to $(\mathbb{H}, L_p)$

$$Hol_p(\Omega, \mathbb{H}) = \{f : \Omega \to \mathbb{H} \mid \overline{\partial}_p f = 0 \text{ on } \Omega\} = Ker \overline{\partial}_p$$

where $\overline{\partial}_p$ is Cauchy-Riemann: $\overline{\partial}_p = \frac{1}{2}\left(d + pJ_p^* \circ d\right)$
For any positive o.n. basis $\{1, p, q, pq\}$ of $\mathbb{H}$ $(p, q \in \mathbb{S}^2)$, the equations of regularity can be rewritten as

$$\overline{\partial}_p f_1 = J_q^* (\partial_p \overline{f}_2)$$

where $f = (f^0 + pf^1) + (f^2 + pf^3)q = f_1 + f_2 q$ are defined by the decomposition $\mathbb{H} = \mathbb{C}_p \oplus (\mathbb{C}_p)q$

$$\Rightarrow \text{ every } f \in Hol_p(\Omega, \mathbb{H}) \text{ is a regular function on } \Omega.$$

Totally regular functions

Definition (cf. *totally analytic variables* - Delanghe 1970)

A regular function $f \in \mathcal{R}(\Omega)$ is totally regular if the powers f^k are regular on Ω for every integer $k \geq 0$ and f^k is regular on $\Omega' = \{x \in \Omega \mid f(x) \neq 0\}$ for every integer $k < 0$.

Example

Every $\mathbb{C}_p$–valued function $f \in Hol_p(\Omega, \mathbb{H})$ is totally regular.

This follows from the isomorphism of commutative algebras

$$Hol_p(\Omega, \mathbb{C}_p) \overset{\phi_a}{\simeq} Hol(\Omega^a, \mathbb{C})$$

defined by composition with 3D–rotations $rot_a(q) = aqa^{-1}$ and $rot_{\gamma(a)}$ with $rot_a(i) = \gamma(p)$ and $\Omega^a = a^{-1}\Omega a$:

$$\phi_a(f) = f^a := rot_{\gamma(a)} \circ f \circ rot_a$$

Totally regular functions

Remark

When $f \in Hol_p(\Omega, \mathbb{H})$ is not $\mathbb{C}_p$–valued, the decomposition $f = f_1 + f_2 q$ w.r.t. any orthonormal basis $\{p, q, pq\}$ defines totally regular components f_1, $f_2 \in Hol_p(\Omega, \mathbb{C}_p)$.

In the *affine* case, $f = \sum_{\alpha=0}^{4} x_\alpha a_\alpha + b$, $a_\alpha, b \in \mathbb{H}$, we can characterize completely the totally regular functions:

Theorem

Every non–constant affine totally regular function belongs to the union $\cup_{p \in \mathbb{S}^2} Hol_p(\mathbb{H}, \mathbb{C}_p)$.

The result follows from the properties:

(1) If f is affine and f, f^2 are regular, then $J(f)^{adj} = 0$, i.e. f has maximum rank 2 (cf. Gürlebeck and Sprössig 1990)

(2) f has maximum rank 2 $\Rightarrow \exists p \in \mathbb{S}^2$ such that $f \in Hol_p(\mathbb{H}, \mathbb{H})$.

Totally regular functions

Property (2) is an application of the energy quadric of a regular function f. It is a family of positive semi–definite quadrics $M(f)$ which contains information about the holomorphicity properties of the function (*Lichnerowicz invariants*). In the affine case,

$$\det M(f) = 0 \Leftrightarrow \exists p : f \in Hol_p(\mathbb{H}, \mathbb{H})$$

and the formula

$$\det M(f) = \frac{1}{2}\left(|e_1|^2 + |e_2|^2\right)$$

holds, with e_1, e_2 linear combinations of elements of $J(f)^{adj}$.

(3) If $f \in Hol_p(\mathbb{H}, \mathbb{H})$ is affine, $rk(f) = 2$ and f^2 is regular, then $f \in Hol_p(\mathbb{H}, \mathbb{C}_p)$.

Corollary

If $f \in \mathcal{R}(\mathbb{H})$ is affine and f^2 is regular, then f is totally regular.

Totally regular functions

In the *affine* case

$$rk(f) \leq 2 \Rightarrow f \text{ is a } J_p\text{–holomorphic map}$$

In the *general* case, we can say something weaker:

Theorem

Let $f \in \mathcal{R}(\Omega)$. Then

- *If $rk(f) \leq 2$, then f is a (pseudo)holomorphic map w.r.t. a (not nec. constant) almost complex structure $p(z)$ defined on a dense subset of Ω.*
- *If $Im(f)$ is contained in a (real) plane H, then there exists $p \in \mathbb{S}^2$ such that $f \in Hol_p(\Omega, \mathbb{H})$.*
- *If $Im(f)$ is contained in $\mathbb{C}_p$ for some $p \in \mathbb{S}^2$ (i.e. $H \supseteq \mathbb{R}$), then f is a J_p–holomorphic function, and therefore it is totally regular.*

Totally regular functions and biregularity

The "twisted" *Fueter variables*

$$v_i = x_0 + x_1 i \in Hol_i(\mathbb{H}, \mathbb{C}_i)$$
$$v_j = x_0 + x_2 j \in Hol_j(\mathbb{H}, \mathbb{C}_j)$$
$$v_k = x_0 - x_3 k \in Hol_k(\mathbb{H}, \mathbb{C}_k)$$

are totally regular. We can define, for any $p \in \mathbb{S}^2$, a totally regular function $v_p \in Hol_p(\mathbb{H}, \mathbb{C}_p)$, which generalizes Fueter variables.

Definition

Let $p \in \mathbb{S}^2$ and $\gamma(p) = p_1 i + p_2 j - p_3 k$. Let $\vec{x}$ denote the vector $(x_1, x_2, x_3) \in \mathbb{R}^3$ defined by $x \in \mathbb{H}$. We set

$$v_p(x) = x_0 + \left(\overrightarrow{\gamma(p)} \cdot \vec{x} \right) p$$

Totally regular functions and biregularity

The function v_p can be seen as one component of a biregular function, that is an invertible regular function $f \in C^1(\Omega)$ such that f^{-1} is regular.

Theorem

- *For every $p \in \mathbb{S}^2$, $v_p \in Hol_p(\mathbb{H}, \mathbb{C}_p) \Rightarrow v_p$ is totally regular.*
- *For any $q \in \mathbb{S}^2$, $q \perp p$, let $a \in \mathbb{H}$ s.t. $rot_{\gamma(a)}(i) = p$, $rot_{\gamma(a)}(j) = q$. There exists an affine biregular function*

$$f_a = v_p + w_a q$$

with totally regular components $v_p, w_a \in Hol_p(\mathbb{H}, \mathbb{C}_p)$. $f_a \in Hol_p(\mathbb{H}, \mathbb{H})$ is J_p–biholomorphic, with inverse

$$f_a^{-1} = f_{a'} \in Hol_{\gamma(p)}(\mathbb{H}, \mathbb{H}) \quad (a' = \gamma(a)^{-1}).$$

Totally regular functions and biregularity

Remark

We can take as $f_a = v_p + w_a q$ the biregular rotation

$$f_a = id^a = rot_{\gamma(a)} \circ id \circ rot_a = rot_{\gamma(a)a}$$

defined by the *reduced quaternion* $\gamma(a)a \;\Rightarrow\; w_a = z_2^a$.

Using the energy quadric of f_a, it can be easily shown that $rot_{\gamma(a)a}$ is holomorphic w.r.t. a circle of structures $p' \in \mathbb{S}^2$:

$$rot_{\gamma(a)a} \in Hol_{p'}(\mathbb{H}, \mathbb{H}) \quad \forall\, p' \in \langle rot_{\gamma(a)}(i), rot_{\gamma(a)}(j)\rangle \cap \mathbb{S}^2$$

Another application of the energy quadric:

Theorem (P. AACA)

An affine biregular function is always J_p–biholomorphic for some p: $\exists p$ s.t. $f \in Hol_p(\mathbb{H}, \mathbb{H})$ and $f^{-1} \in Hol_{\gamma(p)}(\mathbb{H}, \mathbb{H})$.

Totally regular functions and biregularity

Examples

1. $p = i, q = j \Rightarrow a = 1$

$$\Rightarrow f_a = id = z_1 + z_2 j = (x_0 + x_1 i) + (x_2 + x_3 i)j \in Hol_i$$

2. $p = i, q = k \Rightarrow a = \frac{j-k}{\sqrt{2}}$ (f_a depends on the choice of $q \perp p$)

$$\Rightarrow f_a = z_1 - z_2 j = (x_0 + x_1 i) + (-x_3 + x_2 i)k \in Hol_i$$

3. $p = j, q = k \Rightarrow a = \frac{1+i+j-k}{2}$

$$\Rightarrow f_a = \bar{z}_1 + \bar{z}_2 j = (x_0 + x_2 j) - (x_3 + x_1 j)k \in Hol_j$$

4. $p = k, q = j \Rightarrow a = \frac{1-j}{\sqrt{2}}$

$$\Rightarrow f_a = \bar{z}_1 + \bar{z}_2 j = (x_0 - x_3 k) + (x_2 + x_1 k)j \in Hol_k$$

Quaternionic Lagrange Interpolation

Lagrange Interpolation in $\mathbb{H}$:
Given k distinct points $a_1, \dots, a_k \in \mathbb{H}$ and k values $u_1, \dots, u_k \in \mathbb{H}$, find a *Lagrange polynomial* in the module of regular functions, i.e. a polynomial

$$L \in \mathcal{R}(\mathbb{H}) \quad \text{s.t.} \quad L(a_j) = u_j \quad \text{for every } j = 1, \dots, k.$$

(cf. Gürlebeck and Sprössig 1990)

Lemma (1)

Let $p \in \mathbb{S}^2$ be fixed. Given any J_p-biholomorphic ($\Rightarrow$ biregular) mapping $f \in Hol_p(\mathbb{H}, \mathbb{H})$, let $f = f_1 + f_2 q$ ($q \perp p$). There exist (infinitely many) $\alpha, \beta \in \mathbb{C}_p$ s.t.

$$g = \alpha f_1 + \beta f_2 \in Hol_p(\mathbb{H}, \mathbb{C}_p)$$

is totally regular and satisfies the conditions $g(a_i) \neq g(a_j)\ \forall\ i \neq j$. The numbers α, β can also be found in $\mathbb{R}$.

Quaternionic Lagrange Interpolation

From the Lemma: every J_p-biholomorphic mapping f gives rise to (an infinite number of) regular *Lagrange interpolation functions* (*polynomials* if f is a polynomial function), given by the formula

$$L = \sum_{s=1}^{k} \ell_s u_s, \quad \text{where}$$

$$\ell_s(x) = \prod_{t \neq s} (g(x) - g(a_t))(g(a_s) - g(a_t))^{-1} \in Hol_p(\mathbb{H}, \mathbb{C}_p)$$

Properties:

- The powers $(\ell_s)^m$ are regular on $\mathbb{H}$ for every integer $m > 0$ (ℓ_s is totally regular) and $L \in Hol_p(\mathbb{H}, \mathbb{H}) \subseteq \mathcal{R}(\mathbb{H})$.
- The powers of L are regular if also the values u_s belong to the subalgebra $\mathbb{C}_p$. In this case also $L \in Hol_p(\mathbb{H}, \mathbb{C}_p)$ is totally regular.

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Properties:

- $Ker(d\ell_s) \supseteq Ker(dg) \Rightarrow rk(L) \leq 2$ (but can be $Im(L) \not\subseteq H$ plane).
- The average of two (or more) interpolating functions can have $rk > 2$ (and still be in $Hol_p(\mathbb{H}, \mathbb{H})$ and then $rk = 4$).

Quaternionic Lagrange Interpolation

Example

If we take the function

$$id^a = rot_{\gamma(a)} \circ id \circ rot_a = rot_{\gamma(a)a}$$

as J_p–biholomorphic mapping, then g is the linear function

$$g = \alpha f_1 + \beta f_2 = \alpha v_p + \beta w_a = \alpha z_1^a + \beta z_2^a$$

If $\alpha, \beta \in \mathbb{R}$, then $g = rot_{\gamma(a)} \circ (\alpha z_1 + \beta z_2) \circ rot_a$.

Remark

If all the points a_s lie in a complex subspace $\mathbb{C}_{\gamma(p)}$ (in particular, in $\mathbb{R}$), then $rot_a(a_s) \in \mathbb{C}$ and the choice $\alpha = 1, \beta = 0$ is valid. If the values u_s are complex (real), then the formula reduces to the usual complex (real) Lagrange interpolation.

Directional quaternionic Lagrange Interpolation

Lemma (2)

For every finite set of distinct points $\{a_s\}_{s=1,\dots,k}$, $\exists$ (infinite values of) $p \in \mathbb{S}^2$ s.t. the totally regular variable $g = v_p$ satisfies the condition

$$g(a_i) = a_{i0} + (\overrightarrow{\gamma(p)} \cdot \vec{a}_i)p \neq a_{j0} + (\overrightarrow{\gamma(p)} \cdot \vec{a}_j)p = g(a_j)$$

for every $i \neq j$ ($\vec{a}_i$ denotes the vector part of a_i).

"Optimal" choice for $p \in \mathbb{S}^2$? For example, minimise the largest angle defined by

$$\min_{\vec{a}_i \neq \vec{a}_j} \frac{|\ \overrightarrow{\gamma(p)} \cdot (\vec{a}_i - \vec{a}_j)\ |}{|\vec{a}_i - \vec{a}_j|}$$

Examples

Let $a_1 = 0, a_2 = 1, a_3 = i, a_4 = j$ and $u_1 = 0, u_2 = j, u_3 = i, u_4 = k$.

(1) Fix $p = i$ and $f = id$ (cf. Lemma 1). We can choose $(\alpha, \beta) = (1, -1)$ and get

$$g = z_1 - z_2$$

$$\Rightarrow \mathcal{E}_B(\ell_1, \ell_2, \ell_3, \ell_4) = \sqrt{\sum_s \mathcal{E}_B(\ell_s)^2} \approx 20.13 \text{ and } \mathcal{E}_B(L) \approx 7.17$$

(2) Let $p = i$ and $f = id$ as before, but now choose $(\alpha, \beta) = (1, 2)$ and get

$$g = z_1 + 2z_2.$$

The interpolating set $\ell'_1, \ell'_2, \ell'_3, \ell'_4 \in Hol_i(\mathbb{H}, \mathbb{C}_i)$ has energy $\mathcal{E}_B(\ell'_1, \ell'_2, \ell'_3, \ell'_4) \approx 243.30$.
$L' = \sum_s \ell'_s u_s \in Hol_i(\mathbb{H}, \mathbb{H})$ has $rk \leq 2$ and energy $\mathcal{E}_B \approx 231.04$

Let $a_1 = 0, a_2 = 1, a_3 = i, a_4 = j$ and $u_1 = 0, u_2 = j, u_3 = i, u_4 = k$.

(3) We can get a better energy $\mathcal{E}_B(\ell_1, \ell_2, \ell_3, \ell_4) \approx 18.73$ choosing

$$g = 0.77z_1 - 0.64z_2$$

(near to optimal for fixed p and f). Now $\mathcal{E}_B(L) \approx 6.70$

(4) *Choose p following Lemma 2.*
Since $\vec{a}_1 = \vec{a}_2 = 0$, $\vec{a}_3 = (1,0,0)$, $\vec{a}_4 = (0,1,0)$, the "directions" to avoid correspond to $p = i, p = \frac{i+j}{\sqrt{2}}, p = j$.

The choice $p = \frac{-i+j}{\sqrt{2}}$ maximises the cosine $\min_{\vec{a}_i \neq \vec{a}_j} \frac{|\overrightarrow{\gamma(p)} \cdot (\vec{a}_i - \vec{a}_j)|}{|\vec{a}_i - \vec{a}_j|}$

$$\Rightarrow \; g = v_p = x_0 + \frac{1}{2}(x_1 - x_2)i + \frac{1}{2}(-x_1 + x_2)j.$$

Then $\mathcal{E}_B(\ell_1, \ell_2, \ell_3, \ell_4) \approx 12.92$ (optimal for $p \in \mathbb{S}^2$) and $\mathcal{E}_B \approx 7.44$

Let $a_1 = 0, a_2 = 1, a_3 = i, a_4 = j$ and $u_1 = 0, u_2 = j, u_3 = i, u_4 = k$.

(5) The average of the interpolating polynomials defined in (3) and (4) has rank 4 (a.e.), is *regular but not–holomorphic* and has energies

$$\mathcal{E}_B \approx 3.50,\ \mathcal{E}_B(\ell_1, \ell_2, \ell_3, \ell_4) \approx 7.92$$

smaller than those of the two polynomials.

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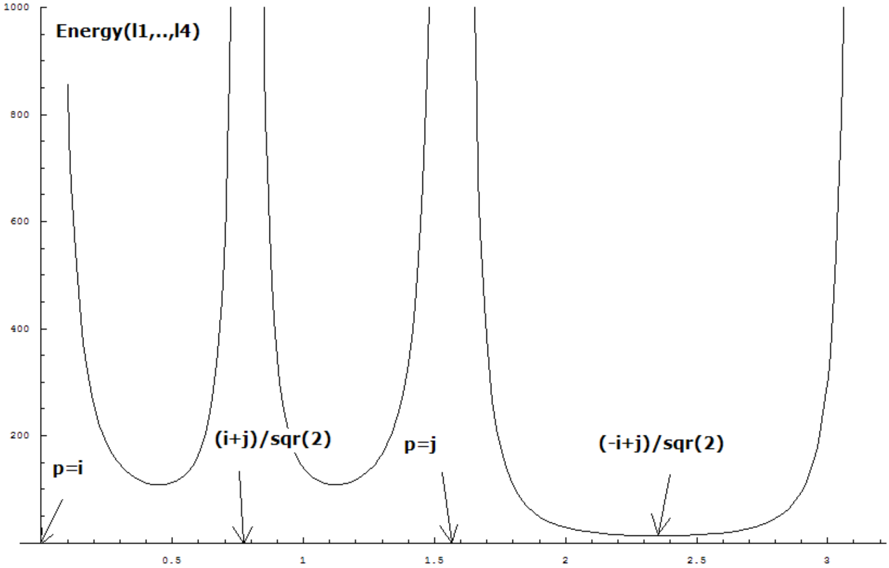
Energy(l1,..,l4)
1000
800
600
400
200
p=i
(i+j)/sqr(2)
p=j
(-i+j)/sqr(2)
0.5
1
1.5
2
2.5
3

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