Boundary values of pluriholomorphic functions in $\mathbb{C}^2$

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We consider the Dirichlet problem for pluriholomorphic functions of two complex variables. Pluriholomorphic functions are solutions of the system $\frac{\partial^2 g}{\partial \bar{z}_i \partial \bar{z}_j} = 0$ for every i, j. The key point is the relation between pluriholomorphic functions and pluriharmonic functions. The link is constituted by the Fueter-regular functions of one quaternionic variable. We apply previous results about the boundary values of pluriharmonic functions and new results on L^2 traces of regular functions to obtain a characterization of the traces of pluriholomorphic functions.

Keywords: Pluriholomorphic functions; Pluriharmonic functions; Quaternionic regular functions.

1. Introduction

We discuss a boundary value problem in two complex variables on a class of pseudoconvex domains containing the unit ball B. The class consists of domains Ω that satisfy a $L^2(\partial\Omega)$-estimate (cf. Sec. 3.2). We conjecture that the estimate holds on every strongly pseudoconvex domain in $\mathbb{C}^2$.

We relate the Dirichlet boundary value problems for pluriholomorphic functions and pluriharmonic functions by means of a class of quaternionic regular functions (cf. Sec. 3), a variant of Fueter-regular functions studied by many authors (see for instance Refs. 11,13,15). Pluriholomorphic functions are solutions of the system $\frac{\partial^2 g}{\partial \bar{z}_i \partial \bar{z}_j} = 0$ for $1 \leq i, j \leq 2$ (see e.g. Refs. 1–4,6–8). The Dirichlet problem for this system is not well posed and the homogeneous problem can have infinitely many independent solutions. As noted in Ref. 8, the Dirichlet problem for pluriharmonic functions has a different character, related to strong ellipticity.

The key point is that if $f = f_1 + f_2 j$ is regular, then f_1 is pluriholomorphic (and harmonic) if and only if f_2 is pluriharmonic.

We begin by giving an application of an existence principle in Functional Analysis proved by Fichera in the 50's. We obtain a result on the boundary values of class $L^2(\partial\Omega)$ of regular functions: every function f_1 which belongs to the class $L^2(\partial\Omega)$ together with its normal derivative $\overline{\partial}_n f_1$ is the first complex component of a regular function on Ω, of class $L^2(\partial\Omega)$. On the unit ball B, where computation of L^2-estimates can be more precise, the result is optimal. We show that the condition on the normal derivative cannot be relaxed and therefore the operation of regular conjugation is not bounded in the harmonic Hardy space $h^2(B)$.

In Sec. 4 we apply the results on the traces of pluriharmonic functions proved in Refs. 5,12 and obtain a characterization of the traces of pluriholomorphic functions. We generalize some results obtained by Detraz[6] and Dzhuraev[7] on the unit ball (cf. also Refs. 1–4,8). We show that if Ω satisfies the $L^2(\partial\Omega)$-estimate, a function $h \in L^2(\partial\Omega)$ with $\overline{\partial}_n h \in L^2(\partial\Omega)$ is the trace of a harmonic pluriholomorphic function on Ω if and only if it satisfies an orthogonality condition (see Theorem 4.2 for the precise statement).

See Ref. 14 for complete proofs of the results presented here.

2. Pluriholomorphic functions

Let Ω be a bounded domain in $\mathbb{C}^2$. A complex valued function $g \in C^2(\Omega)$ is *pluriholomorphic* on Ω if it satisfies the PDE system

$$\frac{\partial^2 g}{\partial \bar{z}_i \partial \bar{z}_j} = 0 \quad \text{on } \Omega \quad (1 \le i, j \le 2).$$

We refer to the works of Detraz,[6] Dzhuraev[7,8] and Begehr[1,2] for properties of pluriholomorphic functions of two or more variables. For the one-variable case there is a vast literature by Balk and his school (in their papers pluriholomorphic functions are called polyanalytic functions of order two). Every function g in the space $Phol(\Omega)$ of pluriholomorphic functions on Ω has a representation of the form: $g = g_0 + \sum_{i=1}^{2} \bar{z}_i g_i$, where g_0, g_1, g_2 are holomorphic on Ω. Note that the g_i's can be less regular than g on $\overline{\Omega}$.

2.1. *Dirichlet problem for pluriholomorphic functions*

Given a continuous function h on $\partial\Omega$, the following boundary problem

$$\begin{cases} g \in C^2(\Omega) \cap C(\overline{\Omega}) \\ \frac{\partial^2 g}{\partial \bar{z}_i \partial \bar{z}_j} = 0 \quad \text{on } \Omega \\ g_{|\partial\Omega} = h \quad \text{on } \partial\Omega \end{cases}$$

is not well posed (also for $n = 1$). Several facts about this problem can be found in the works of Begehr and Dzhuraev.[1–4] For instance, on the unit ball B the homogeneous problem with boundary datum $h = 0$ has infinitely many independent solutions $u_k(z) = (|z_1|^2+|z_2|^2-1)z_1^{k_1}z_2^{k_2}$, $|k| = k_1 + k_2 \geq 0$. Detraz[6] proved that the nonhomogeneous problem imposes compatibility conditions on h: if $S = \partial B$ and $h \in C^2(S)$, then h must satisfy the tangential equation $LLh = 0$ on S, where $L = z_2\frac{\partial}{\partial \bar{z}_1} - z_1\frac{\partial}{\partial \bar{z}_2}$.

3. Fueter regular functions

We identify the space $\mathbb{C}^2$ with the set $\mathbb{H}$ of quaternions by means of the mapping that associates the pair $(z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$ with the quaternion $q = z_1+z_2j = x_0+ix_1+jx_2+kx_3 \in \mathbb{H}$. A quaternionic function $f = f_1 + f_2j \in C^1(\Omega)$ is *(left) regular* (or *hyperholomorphic*) on Ω if

$$\mathcal{D}f = 2\left(\frac{\partial}{\partial \bar{z}_1} + j\frac{\partial}{\partial \bar{z}_2}\right) = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} - k\frac{\partial f}{\partial x_3} = 0 \quad \text{on } \Omega.$$

With respect to this definition of regularity, the space $\mathcal{R}(\Omega)$ of regular functions contains the identity mapping and every holomorphic mapping (f_1, f_2) on Ω (w.r.t. the standard complex structure) defines a regular function $f = f_1 + f_2j$. We recall some properties of regular functions, for which we refer to the papers of Sudbery,[16] Shapiro and Vasilevski[15] and Nōno:[11]

(1) The complex components are both holomorphic or both non-holomorphic.
(2) Every regular function is harmonic.
(3) If Ω is pseudoconvex, every complex harmonic function is the complex component of a regular function on Ω.
(4) The space $\mathcal{R}(\Omega)$ of regular functions on Ω is a *right* $\mathbb{H}$-module with integral representation formulas.

3.1. *A link between pluriholomorphic functions and pluriharmonic functions*

In terms of its complex components f_1 and f_2 (called quaternionic *conjugate harmonic* functions), the regularity of $f = f_1 + f_2j$ is equivalent to

$$\frac{\partial f_1}{\partial \bar{z}_1} = \frac{\partial \overline{f_2}}{\partial z_2}, \quad \frac{\partial f_1}{\partial \bar{z}_2} = -\frac{\partial \overline{f_2}}{\partial z_1}.$$

It follows easily that if $f = f_1 + f_2j$ is regular, then f_1 is pluriholomorphic (and harmonic) if, and only if, f_2 is pluriharmonic. i.e. $\partial\overline{\partial} f_2 = 0$ on Ω.

The Dirichlet problem for pluriharmonic functions

$$\begin{cases} g \in C^2(\Omega) \cap C(\overline{\Omega}) \\ \frac{\partial^2 g}{\partial z_i \partial \bar{z}_j} = 0 \quad \text{on } \Omega \\ g_{|\partial\Omega} = h \quad \text{on } \partial\Omega \end{cases}$$

is characterized by strong ellipticity. The solution, if it exists, is unique, and the system can be splitted into equations for the real and imaginary parts of g.

3.2. *Quaternionic harmonic conjugation*

In the following we shall assume that the domain Ω satisfies the $L^2(\partial\Omega)$-estimate

$$|(f, Lg)| \leq C\|\partial_n f\|\|\overline{\partial}_n g\| \tag{1}$$

for every complex harmonic functions f, g on Ω, of class C^1 on $\overline{\Omega}$, where L is the tangential Cauchy-Riemann operator

$$L = \frac{1}{|\partial\rho|}\left(\frac{\partial\rho}{\partial\bar{z}_2}\frac{\partial}{\partial\bar{z}_1} - \frac{\partial\rho}{\partial\bar{z}_1}\frac{\partial}{\partial\bar{z}_2}\right),$$

(ρ a defining function for Ω) $\overline{\partial}_n f$ is the normal part of $\overline{\partial} f$ on $\partial\Omega$:

$$\overline{\partial}_n f = \sum_k \frac{\partial f}{\partial\bar{z}_k}\frac{\partial\rho}{\partial z_k}\frac{1}{|\overline{\partial}\rho|} \quad \text{or} \quad \overline{\partial}_n f d\sigma = *\overline{\partial} f_{|\partial\Omega}$$

and $\partial_n f$ is the normal part of ∂f. Here $*$ is the Hodge operator on forms.

Theorem 3.1.

(i) On the unit ball B, the estimate (1) is satisfied with constant $C = 1$.
(ii) If Ω satisfies the estimate (1), then it is a domain of holomorphy.

We conjecture that the estimate holds for every (strongly) pseudoconvex domain in $\mathbb{C}^2$. We now prove a result about the existence of a quaternionic conjugate harmonic in the space $L^2(\partial\Omega)$. We use the following Hilbert subspace of complex valued functions in $L^2(\partial\Omega)$:

$$\overline{W}^1_n(\partial\Omega) = \{f \in L^2(\partial\Omega) \mid \overline{\partial}_n f \in L^2(\partial\Omega)\}$$

with product $(f, g)_{\overline{W}^1_n} = (f, g) + (\overline{\partial}_n f, \overline{\partial}_n g)$.

Theorem 3.2. *Assume $\partial\Omega$ connected. Given $f_1 \in \overline{W}^1_n(\partial\Omega)$, there exists $f_2 \in L^2(\partial\Omega)$ (unique up to a CR function) such that $f = f_1 + f_2 j$ is the trace of a regular function on Ω. Moreover, f_2 satisfies the estimate*

$$\inf_{f_0} \|f_2 + f_0\|_{L^2(\partial\Omega)} \leq C\|f_1\|_{\overline{W}^1_n(\partial\Omega)},$$

where the infimum is taken among the CR functions $f_0 \in L^2(\partial\Omega)$.

On the unit ball B, a sharper estimate can be proved.

Theorem 3.3. *Given $f_1 \in \overline{W}^1_n(S)$, there exists $f_2 \in L^2(S)$ (unique up to a CR function) such that $f = f_1 + f_2 j$ is the trace of a regular function on B. Moreover, f_2 satisfies the estimate*

$$\inf_{f_0 \in CR(S)} \|f_2 + f_0\|_{L^2(S)} \leq \|\overline{\partial}_n f_1\|_{L^2(S)}.$$

Remark 3.1. The condition on f_1 cannot be relaxed: $\exists f_1 \in L^2(S)$ for which it does not exist any $L^2(S)$ function f_2 such that $f_1 + f_2 j$ is the trace of a regular function on B. This means that the operation of quaternionic regular conjugation is not bounded in the harmonic Hardy space $h^2(B)$. This is different from pluriharmonic conjugation (cf. Stout[17]) and in particular from the (complex) one-variable situation.

Example 3.1. A function $f_1 \in L^2(S)\backslash\overline{W}^1_n(S)$ with the required properties is $f_1 = z_2(1 - \bar{z}_1)^{-1}$.

4. Boundary values

We recall a characterization of the boundary values of pluriharmonic functions, proposed by Fichera in the 1980's and proved in Refs. 5 and 12. Let Ω have connected boundary of class C^1. Let

$$Harm^1_0(\Omega) = \{\phi \in C^1(\overline{\Omega}) \mid \phi \text{ is harmonic on } \Omega,\ \overline{\partial}_n\phi \text{ is real on } \partial\Omega\}.$$

This space can be characterized by means of the Bochner-Martinelli operator of the domain Ω. Cialdea[5] proved the following result for boundary values of class L^2 (and more generally of class L^p).

Theorem 4.1. *Let $g \in L^2(\partial\Omega)$ be complex valued. Then g is the trace of a pluriharmonic function on Ω if and only if the following orthogonality condition is satisfied:*

$$\int_{\partial\Omega} g * \overline{\partial}\phi = 0 \quad \forall\phi \in Harm^1_0(\Omega).$$

4.1. *Boundary values of pluriholomorphic functions*

If $f \in C^1(\overline{\Omega})$ is regular, on the boundary $\partial\Omega$ it satisfies the equations $\overline{\partial}_n f_1 = -\overline{L(f_2)}$, $\overline{\partial}_n f_2 = \overline{L(f_1)}$. More generally, if $f = f_1 + f_2 j : \partial\Omega \to \mathbb{H}$ is a function of class $L^2(\partial\Omega)$ and it is the trace of a regular function on Ω, then it satisfies the integral condition

$$\int_{\partial\Omega} f_1\, \overline{\partial}\phi \wedge d\zeta = -2 \int_{\partial\Omega} \overline{f_2} * \overline{\partial}\phi \quad \forall \phi \in Harm^1(\Omega).$$

If $\partial\Omega$ is connected, it can be proved that also the converse is true. As a corollary, we get the following result.

Theorem 4.2. *Assume that Ω has connected boundary and satisfies the $L^2(\partial\Omega)$-estimate. Let $h \in \overline{W}^1_n(\partial\Omega)$. Then h is the trace of a harmonic pluriholomorphic function on Ω if and only if the following orthogonality condition is satisfied:*

$$\int_{\partial\Omega} h\, \overline{\partial}\phi \wedge d\zeta = 0 \quad \forall \phi \in Harm^1_0(\Omega). \tag{2}$$

On the unit ball B we can get a more precise result:

(1) If $h \in Phol(B) \cap C^1(\overline{B})$, then Lh is CR on S.
(2) If $h \in Phol(B) \cap C^2(\overline{B})$, then $LLh = 0$ on S.
(3) If $h \in C^1(S)$ and h is the trace on S of a pluriholomorphic function on $B \Rightarrow Lh \in CR(S) \Rightarrow$ condition (2) is satisfied.
(4) If h is of class $C^{1+\alpha}(S)$, with $\alpha > 0$, the conditions in (3) are all equivalent.

In particular, we get Detraz's result for C^2 functions: h is the trace on S of a pluriholomorphic function on B if and only if $LLh = 0$ on S.

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