Slice Functional Calculus in Quaternionic Hilbert Spaces

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Abstract. We propose a continuous functional calculus in quaternionic Hilbert spaces. The class of continuous functions considered is the one of slice quaternionic functions. Slice functions generalize the concept of slice regular function, which comprises power series with quaternionic coefficients on one side and that can be seen a generalization to quaternons of holomorphic functions of one complex variable. The notion of slice function allows to introduce suitable classes of real, complex and quaternionic C^*–algebras and to define, on each of these C^*–algebras, a functional calculus for quaternionic normal operators.

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1. Introduction

We start from basic issues regarding the general notion of *spherical spectrum* of an operator on a (right) quaternionic Hilbert space. For the definition of the spectrum we follow the viewpoint adopted in [3] for quaternionic Banach modules. A pivotal tool in our investigation is the notion of *slice function* [8]. That notion allows one to introduce suitable classes of real, complex and quaternionic C^*–algebras of functions and to define, on each of these C^*–algebras, a functional calculus for normal operators. In particular, we establish several versions of the *spectral map theorem*. For quaternionic Hilbert spaces, a formulation of the spectral theorem already exists [9] without any systematic investigation of the continuous functional calculus. We also show that our continuous functional calculus, when restricted to slice regular functions, coincides with the functional calculus developed in [3] as a generalization of the classical holomorphic functional calculus. We refer to [6] for complete proofs of the stated result.

2. Quaternionic Hilbert spaces

We recall some basic notions about quaternionic Hilbert spaces (see e.g. [1]). Let $\mathbb{H}$ denote the skew field of quaternions. Let H be a *right* $\mathbb{H}$–module. H is called a *quaternionic pre–Hilbert space* if there exists a Hermitian quaternionic scalar product $\mathsf{H} \times \mathsf{H} \ni (u, v) \mapsto \langle u|v\rangle \in \mathbb{H}$ satisfying the following three properties:

- *Right linearity*: $\langle u|vp + wq\rangle = \langle u|v\rangle p + \langle u|w\rangle q$ if $p, q \in \mathbb{H}$ and $u, v, w \in \mathsf{H}$.
- *Quaternionic Hermiticity*: $\langle u|v\rangle = \overline{\langle v|u\rangle}$ if $u, v \in \mathsf{H}$.
- *Positivity*: If $u \in \mathsf{H}$, then $\langle u|u\rangle \in \mathbb{R}^+$ and $u = 0$ if $\langle u|u\rangle = 0$.

We can define the *quaternionic norm* by setting

$$\|u\| := \sqrt{\langle u|u\rangle} \in \mathbb{R}^+ \quad \text{if } u \in \mathsf{H}.$$

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Definition 2.1. A quaternionic pre–Hilbert space H is said to be a *quaternionic Hilbert space* if it is complete with respect to its natural distance $d(u,v) := \|u-v\|$.

Example. The space $\mathbb{H}^n$ with scalar product $\langle u, v\rangle = \sum_{i=1}^n \bar{u}_i v_i$ is a finite-dimensional quaternionic Hilbert space.

Definition 2.2. A *right* $\mathbb{H}$*–linear operator* is a map $T : D(T) \longrightarrow \mathsf{H}$ such that:

$$T(ua+vb) = (Tu)a + (Tv)b \quad \text{if } u,v \in D(T) \text{ and } a,b \in \mathbb{H},$$

where the *domain* $D(T)$ of T is a (not necessarily closed) right $\mathbb{H}$–linear subspace of H.

It can be shown that an operator $T : D(T) \longrightarrow \mathsf{H}$ is continuous if and only if it is bounded, i.e. there exists $K \geq 0$ such that

$$\|Tu\| \leq K\|u\| \quad \text{for each } u \in D(T).$$

The set $\mathfrak{B}(\mathsf{H})$ of all bounded operators $T : \mathsf{H} \longrightarrow \mathsf{H}$ is a complete metric space w.r.t. the metric $D(T,S) := \|T-S\|$, where $\|T\| := \sup_{u\in D(T)\setminus\{0\}} \frac{\|Tu\|}{\|u\|} = \inf\{K \in \mathbb{R} \mid \|Tu\| \leq K\|u\| \ \forall u \in D(T)\}$.

Many assertions that are valid in the complex Hilbert spaces case, continue to hold for quaternionic operators. We mention the uniform boundedness principle, the open map theorem, the closed graph theorem, the Riesz representation theorem and the polar decomposition of operators.

Left scalar multiplications. It is possible to equip a (right) quaternionic Hilbert space H with a *left* multiplication by quaternions. It is a non–canonical operation relying upon a choice of a preferred Hilbert basis. So, pick out a Hilbert basis $\mathcal{N}$ of H and define the *left scalar multiplication of* H *induced by* $\mathcal{N}$ as the map $\mathbb{H} \times \mathsf{H} \ni (q,u) \mapsto qu \in \mathsf{H}$ given by

$$qu := \textstyle\sum_{z\in\mathcal{N}} zq\langle z|u\rangle \quad \text{if } u \in \mathsf{H} \text{ and } q \in \mathbb{H}.$$

For every $q \in \mathbb{H}$, the map $L_q : u \mapsto qu$ belongs to $\mathfrak{B}(\mathsf{H})$. Moreover, the map $\mathcal{L}_\mathcal{N} : \mathbb{H} \longrightarrow \mathfrak{B}(\mathsf{H})$, defined by setting $\mathcal{L}_\mathcal{N}(q) := L_q$ is a norm–preserving real algebra homomorphism.

The set $\mathfrak{B}(\mathsf{H})$ is always a *real Banach* C^**–algebra with unity.* It suffices to consider the right scalar multiplication $(Tr)(u) = T(u)r$ for real r and the adjunction $T \mapsto T^*$ as *–involution. By means of a left scalar multiplication, it can be given the richer structure of *quaternionic Banach* C^**–algebra.*

Theorem 2.1. *Let* H *be a quaternionic Hilbert space equipped with a left scalar multiplication. Then the set* $\mathfrak{B}(\mathsf{H})$*, equipped with the pointwise sum, with the scalar multiplications defined by*

$$(qT)u := q(Tu) \quad \textit{and} \quad (Tq)(u) := T(qu),$$

with the composition as product, with $T \mapsto T^*$ *as* **-involution, is a* quaternionic two–sided Banach C^*–algebra with unity*.*

Observe that the map $\mathcal{L}_\mathcal{N}$ gives a *-representation of $\mathbb{H}$ in $\mathfrak{B}(\mathsf{H})$.

3. Resolvent and spectrum

It is not clear how to extend the definitions of spectrum and resolvent in quaternionic Hilbert spaces. Let us focus on the simpler case of eigenvalues of a bounded right $\mathbb{H}$–linear operator T. Without fixing any left scalar multiplication of H, the equation determining the eigenvalues reads as follows:

$$Tu = uq.$$

Here a drawback arises: if $q \in \mathbb{H} \setminus \mathbb{R}$ is fixed, the map $u \mapsto uq$ is not right $\mathbb{H}$–linear. Consequently, the eigenspace of q cannot be a right $\mathbb{H}$–linear subspace. Indeed, if $\lambda \neq 0$, $u\lambda$ is an eigenvector of $\lambda^{-1}q\lambda$ instead of q itself. As a second guess, one could decide to deal with quaternionic Hilbert spaces equipped with a left scalar multiplication and require that

$$Tu = qu.$$

Now both sides are right $\mathbb{H}$–linear. However, this approach is not suitable for physical applications, where self–adjoint operators should have real spectrum. We come back to the former approach and accept that each eigenvalue q brings a whole conjugation class of the quaternions, the *eigensphere*

$$\mathbb{S}_q := \{\lambda^{-1}q\lambda \in \mathbb{H} \mid \lambda \in \mathbb{H} \setminus \{0\}\}.$$

We adopt the viewpoint introduced in [3] for quaternionic two-sided Banach modules. Given an operator $T : D(T) \longrightarrow \mathsf{H}$ and $q \in \mathbb{H}$, let

$$\Delta_q(T) := T^2 - T(q + \overline{q}) + \mathrm{I}|q|^2.$$

Definition 3.1. The *spherical resolvent set* of T is the set $\rho_S(T)$ of $q \in \mathbb{H}$ such that:
(a) $Ker(\Delta_q(T)) = \{0\}$.
(b) $Range(\Delta_q(T))$ is dense in H.
(c) $\Delta_q(T)^{-1} : Range(\Delta_q(T)) \longrightarrow D(T^2)$ is bounded.

The *spherical spectrum* $\sigma_S(T)$ of T is defined by $\sigma_S(T) := \mathbb{H} \setminus \rho_S(T)$. It decomposes into three disjoint *circular* (i.e. invariant by conjugation) subsets:
(i) the *spherical point spectrum* of T (the set of *eigenvalues*):

$$\sigma_{pS}(T) := \{q \in \mathbb{H} \mid Ker(\Delta_q(T)) \neq \{0\}\}.$$

(ii) the *spherical residual spectrum* of T:

$$\sigma_{rS}(T) := \left\{q \in \mathbb{H} \,\middle|\, Ker(\Delta_q(T)) = \{0\},\ \overline{Range(\Delta_q(T))} \neq \mathsf{H} \right\}.$$

(iii) the *spherical continuous spectrum* of T:

$$\sigma_{cS}(T) := \left\{q \in \mathbb{H} \,\middle|\, \Delta_q(T)^{-1} \text{ is densely defined but not bounded} \right\}.$$

The *spherical spectral radius* of T is defined as

$$r_S(T) := \sup\left\{|q| \,\middle|\, q \in \sigma_S(T)\right\} \in \mathbb{R}^+ \cup \{+\infty\}.$$

In our context, the subspace $Ker(\Delta_q(T))$ has the role of an "eigenspace". In particular, $Ker(\Delta_q(T)) \neq \{0\}$ if and only if $\mathbb{S}_q$ is an eigensphere of T.

3.1. Spectral properties

The spherical resolvent and the spherical spectrum can be defined for bounded right $\mathbb{H}$–linear operators on quaternionic two–sided Banach modules in a form similar to that introduced above (see [3]). Several spectral properties of bounded operators on complex Banach or Hilbert spaces remain valid in that general context. Here we recall some of these properties in the quaternionic Hilbert setting.

Theorem 3.1. *Let* H *be a quaternionic Hilbert space and let* $T \in \mathfrak{B}(\mathsf{H})$. *Then*
(a) $r_S(T) \leq \|T\|$.
(b) $\sigma_S(T)$ *is a non–empty compact subset of* $\mathbb{H}$.
(c) *Let* $P \in \mathbb{R}[X]$. *Then, if* T *is self–adjoint, the following* spectral map property *holds:*

$$\sigma_S(P(T)) = P(\sigma_S(T)).$$

(d) Gelfand's spectral radius formula *holds:*

$$r_S(T) = \lim_{n \to +\infty} \|T^n\|^{1/n}.$$

In particular, if T *is* normal *(i.e.* $TT^* = T^*T$*), then* $r_S(T) = \|T\|$.

Regardless different definitions with respect to the complex Hilbert space case, the notions of spherical spectrum and resolvent set enjoy some properties which are quite similar to those for complex Hilbert spaces. Other features, conversely, are proper to the quaternionic Hilbert space case. First of all, it turns out that the spherical point spectrum coincides with the set of eigenvalues of T.

Proposition 3.2. *Let* H *be a quaternionic Hilbert space and let* $T : D(T) \longrightarrow \mathsf{H}$ *be an operator. Then* $\sigma_{pS}(T)$ *coincides with the set of all eigenvalues of* T.

Theorem 3.2. *Let T be an operator with dense domain on a quaternionic Hilbert space H.*

(a) $\sigma_S(T) = \sigma_S(T^*)$.
(b) *If $T \in \mathfrak{B}(\mathsf{H})$ is normal, then*
  (i) $\sigma_{pS}(T) = \sigma_{pS}(T^*)$.
  (ii) $\sigma_{rS}(T) = \sigma_{rS}(T^*) = \emptyset$.
  (iii) $\sigma_{cS}(T) = \sigma_{cS}(T^*)$.
(c) *If T is self–adjoint, then $\sigma_S(T) \subset \mathbb{R}$ and $\sigma_{rS}(T)$ is empty.*
(d) *If T is anti self–adjoint, then $\sigma_S(T) \subset \mathrm{Im}(\mathbb{H})$ and $\sigma_{rS}(T)$ is empty.*
(e) *If $T \in \mathfrak{B}(\mathsf{H})$ is unitary, then $\sigma_S(T) \subset \{q \in \mathbb{H} \,|\, |q| = 1\}$.*
(f) *If $T \in \mathfrak{B}(\mathsf{H})$ is anti self–adjoint and unitary, then $\sigma_S(T) = \sigma_{pS}(T) = \mathbb{S}$ (the sphere of quaternionic imaginary units).*

It can be shown that, differently from operators on complex Hilbert spaces, a normal operator T on a quaternionic space is unitarily equivalent to T^*.

4. Slice functions

The concept of *slice regularity* has been introduced by Gentili and Struppa [4, 5] for functions of one quaternionic variable and then extended to other real *-algebras (e.g. Clifford algebras [2] and alternative *–algebras [7, 8]). This function theory comprises polynomials and power series with quaternionic coefficients on one side. At the base of the definition there is the "slice" character of $\mathbb{H}$:

- $\mathbb{H} = \bigcup_{\jmath \in \mathbb{S}} \mathbb{C}_\jmath$ where $\mathbb{C}_\jmath$ is the real subalgebra $\langle \jmath \rangle \simeq \mathbb{C}$.
- $\mathbb{C}_\jmath \cap \mathbb{C}_\kappa = \mathbb{R}$ for every $\jmath, \kappa \in \mathbb{S}$ with $\jmath \neq \pm\kappa$.

The original definition requires that, for every $\jmath \in \mathbb{S}$, the restriction $f_{|\mathbb{C}_\jmath}$ is holomorphic with respect to the complex structures given by left multiplication by $\jmath$. Another approach (see [7, 8]) starts from the embedding of the space of slice regular functions into a larger class, that of continuous *slice functions*. Given $\mathcal{K} \subset \mathbb{C}$, consider the *circular set* $\Omega_\mathcal{K}$ defined by

$$\Omega_\mathcal{K} = \{\alpha + \jmath\beta \in \mathbb{H} \,|\, \alpha, \beta \in \mathbb{R},\, \alpha + i\beta \in \mathcal{K},\, \jmath \in \mathbb{S}\}.$$

Let $\mathbb{H} \otimes_\mathbb{R} \mathbb{C} = \{x + iy \,|\, x, y \in \mathbb{H}\}$, with complex conjugation $w = x + iy \mapsto \overline{w} = x - iy$. A function $F : \mathcal{K} \longrightarrow \mathbb{H} \otimes_\mathbb{R} \mathbb{C}$ satisfying $F(\overline{z}) = \overline{F(z)}$ for every $z \in \mathcal{K}$, is called a *stem function* on $\mathcal{K}$. Any stem function induces a (left) *slice function* $f = \mathcal{I}(F) : \Omega_\mathcal{K} \to \mathbb{H}$: if $q = \alpha + \jmath\beta \in \Omega_\mathcal{K} \cap \mathbb{C}_\jmath$, with $\jmath \in \mathbb{S}$, we set

$$f(q) := F_1(\alpha + i\beta) + \jmath F_2(\alpha + i\beta)$$

where F_1, F_2 are the two $\mathbb{H}$–valued components of F. A quaternionic function f turns out to be *slice regular* if and only if it is the slice function induced by a *holomorphic* stem function F.

4.1. C^*–algebras of slice functions

Given two slice functions $f = \mathcal{I}(F)$ and $g = \mathcal{I}(G)$ on $\Omega_\mathcal{K}$, their pointwise product is not necessarily a slice function. However, we can define their *slice product* by means of the multiplication in $\mathbb{H} \otimes \mathbb{C}$:

$$f \cdot g = \mathcal{I}(FG) = \mathcal{I}((F_1G_1 - F_2G_2) + i(F_1G_2 + F_2G_1)).$$

Theorem 4.1. *The set $\mathcal{S}(\Omega_\mathcal{K}, \mathbb{H})$ of continuous slice functions on $\Omega_\mathcal{K}$ is a quaternionic two–sided Banach C^*–algebra with unity the constant function $1_{\Omega_\mathcal{K}}$ w.r.t. the slice product, the *-involution defined by $f^* := \mathcal{I}(\overline{F_1} - i\overline{F_2})$ and the supremum norm.*

Remark 4.1. The scalar multiplications on $\mathcal{S}(\Omega_\mathcal{K}, \mathbb{H})$ are defined by $f \cdot q := f \cdot c_q$ and $q \cdot f := c_q \cdot f$, where c_q denotes the constant slice function with value q on $\Omega_\mathcal{K}$. $f \cdot q$ coincides with the pointwise scalar multiplication fq for every $q \in \mathbb{H}$. If $q \in \mathbb{R}$, then also $q \cdot f$ is equal to the pointwise scalar multiplication qf. Otherwise, qf is not, in general, a slice function and hence is different from $q \cdot f$.

5. Slice Functional Calculus

5.1. Slice nature of normal operators

The definition of a continuous slice function of a normal operator on a quaternionic Hilbert space is based on the "operatorial" counterpart of the slice character of $\mathbb{H}$.

Theorem 5.1. *Given any normal operator $T \in \mathfrak{B}(\mathsf{H})$, there exist three operators $A, B, J \in \mathfrak{B}(\mathsf{H})$ such that:*

(i) $T = A + JB$.
(ii) *A is self–adjoint and B is positive.*
(iii) *J is anti self–adjoint and unitary.*
(iv) *A, B and J commute mutually.*

Furthermore, it holds:

- *A and B are uniquely determined by T: $A = (T + T^*)\frac{1}{2}$ and $B = |T - T^*|\frac{1}{2}$.*
- *J is uniquely determined by T on $Ker(T - T^*)^{\perp}$.*

(where for $S \in \mathfrak{B}(\mathsf{H})$, $|S|$ denotes the operator defined as the square root of the positive operator S^*S).

This parallelism suggests a natural way to define the operator $f(T)$ for the class of $\mathbb{H}$-*intrinsic* continuous slice functions, i.e. functions satisfying $f(\bar{q}) = \overline{f(q)}$ for every $q \in \Omega_{\mathcal{K}}$ or, equivalently, $f(\Omega_{\mathcal{K}} \cap \mathbb{C}_\jmath) \subset \mathbb{C}_\jmath$ $\forall \jmath \in \mathbb{S}$. If $f = \mathcal{I}(F) = \mathcal{I}(F_1 + iF_2)$ is a *polynomial* slice function, with components F_1, $F_2 \in \mathbb{R}[X, Y]$, we define the normal operator $f(T) \in \mathfrak{B}(\mathsf{H})$ by setting

$$f(T) := F_1(A, B) + JF_2(A, B)$$

and then extend the definition to $\mathbb{H}$–intrinsic *continuous* slice functions on $\sigma_S(T)$ by density.

Remark 5.1.

(i) f is $\mathbb{H}$-intrinsic if and only if the components F_1, F_2 of the stem function F are real valued.
(ii) Even when J is not uniquely determined, $f(T)$ does *not* depend on the choice of the operator J.

Continuous slice functional calculus for normal operators: the $\mathbb{H}$-intrinsic functions case. Consider the commutative real Banach C^*-subalgebra $\mathcal{S}_{\mathbb{R}}(\sigma_S(T), \mathbb{H})$ of $\mathcal{S}(\sigma_S(T), \mathbb{H})$ consisting of $\mathbb{H}$-intrinsic slice functions. The functional calculus $f \mapsto f(T)$ defined above has the following properties.

Theorem 5.2. *The mapping $f \mapsto f(T)$ is the unique continuous *-homomorphism*

$$\Psi_{\mathbb{R},T} : \mathcal{S}_{\mathbb{R}}(\sigma_S(T), \mathbb{H}) \to \mathfrak{B}(\mathsf{H})$$

of real Banach unital C^*-algebras such that:

(i) *$\Psi_{\mathbb{R},T}$ is unity–preserving; that is, $\Psi_{\mathbb{R},T}(1_{\sigma_S(T)}) = \mathrm{I}$.*
(ii) $\Psi_{\mathbb{R},T}(id) = T$.

Moreover, the following facts hold true:

(a) *$f(T)$ is normal.*
(b) *$\Psi_{\mathbb{R},T}$ is isometric: $\|f(T)\| = \|f\|_\infty$ for every $f \in \mathcal{S}_{\mathbb{R}}(\sigma_S(T), \mathbb{H})$.*
(c) *the* spectral map property *$\sigma_S(f(T)) = f(\sigma_S(T))$ holds.*

Continuous slice functional calculus for normal operators: the $\mathbb{C}_\jmath$-slice functions case. The definition of $f(T)$ can be extended to other classes of continuous slice functions. The set $\mathcal{S}_{\mathbb{C}_\jmath}(\Omega_{\mathcal{K}}, \mathbb{H})$ of functions which leave only *one* slice $\mathbb{C}_\jmath$ invariant is a commutative $\mathbb{C}_\jmath$–Banach unital C^*-subalgebra of $\mathcal{S}(\Omega_{\mathcal{K}}, \mathbb{H})$. The space $\mathfrak{B}(\mathsf{H})$ has a similar structure of complex C^*–algebra depending on the choice of the anti self-adjoint operator J such that $T = A + JB$ and J commutes with T, T^*.

Theorem 5.3. *There exists a unique continuous *-homomorphism*

$$\Psi_{\mathbb{C}_\jmath,T} : \mathcal{S}_{\mathbb{C}_\jmath}(\sigma_S(T), \mathbb{H}) \to \mathfrak{B}(\mathsf{H})$$

of $\mathbb{C}_\jmath$–Banach C^*-algebras such that

(i) *$\Psi_{\mathbb{C}_\jmath,T}$ is unity–preserving; that is, $\Psi_{\mathbb{R},T}(1_{\sigma_S(T)}) = \mathrm{I}$.*

(ii) $\Psi_{\mathbb{C}_\jmath,T}(id) = T$.

Moreover, the following facts hold true:

(a) *$f(T)$ is normal.*
(b) *For every $f \in \mathcal{S}_{\mathbb{C}_\jmath}(\sigma_S(T),\mathbb{H})$, the following* $\mathbb{C}_\jmath$–slice spectral map property *holds:*

$$\sigma_S(f(T)) = \Omega_{f(\sigma_S(T)\cap\mathbb{C}_\jmath^+)}$$

(c) *$\Psi_{\mathbb{C}_\jmath,T}$ is norm decreasing: $\|f(T)\| \le \|f\|_\infty$ if $f \in \mathcal{S}_{\mathbb{C}_\jmath}(\sigma_S(T),\mathbb{H})$. More precisely, it holds:*

$$\|f(T)\| = \|f|_{\sigma_S(T)\cap\mathbb{C}_\jmath^+}\|_\infty$$

for every $f \in \mathcal{S}_{\mathbb{C}_\jmath}(\sigma_S(T),\mathbb{H})$.

Continuous slice functional calculus for normal operators: the circular case. Fix an anti self–adjoint and unitary operator $J \in \mathfrak{B}(\mathsf{H})$ such that $T = A + JB$ and J commutes with T, T^*. Choose $\jmath \in \mathbb{S}$ and a left scalar multiplication $q \mapsto L_q$ with $L_\jmath = J$ and $L_q A = A L_q$ and $L_q B = B L_q$ for each $q \in \mathbb{H}$. Then $\mathfrak{B}(\mathsf{H})$ get a structure of quaternionic two–sided Banach unital C^*-algebra.

The set $\mathcal{S}_c(\Omega_\mathcal{K},\mathbb{H})$ of *circular* slice functions, those which satisfy the condition $f(\bar{q}) = f(q)$ for every q, is a non–commutative quaternionic Banach C^*-subalgebra of $\mathcal{S}(\Omega_\mathcal{K},\mathbb{H})$.

Theorem 5.4. *There exists a unique continuous (isometric) *-homomorphism*

$$\Psi_{c,T} : \mathcal{S}_c(\sigma_S(T),\mathbb{H}) \to \mathfrak{B}(\mathsf{H})$$

of quaternionic Banach C^*-algebras such that

(i) *$\Psi_{c,T}$ is unity–preserving; that is, $\Psi_{\mathbb{R},T}(1_{\sigma_S(T)}) = \mathrm{I}$.*
(ii) *$\Psi_{c,T}(id) = T$.*

Continuous slice functional calculus for normal operators: the general case. The previous definitions of $f(T)$ can be extended to a generic continuous slice function $f \in \mathcal{S}(\sigma_S(T),\mathbb{H})$. We get a map $f \mapsto f(T)$ that is $\mathbb{R}$–linear and continuous: there exists $C > 0$ such that

$$\|f(T)\| \le C\|f\|_\infty$$

for every $f \in \mathcal{S}(\sigma_S(T),\mathbb{H})$. In the general case the *-homomorphism property is necessarily lost. However, if e.g. $f \in \mathcal{S}_{\mathbb{C}_\jmath}(\sigma_S(T),\mathbb{H})$ or $g \in \mathcal{S}_c(\sigma_S(T),\mathbb{H})$, then the multiplicative property

$$(f \cdot g)(T) = f(T)g(T)$$

remains true.

5.2. The slice regular case

A functional calculus for slice regular functions of a bounded operator T on a quaternionic two–sided Banach module V has been developed in [3] as a generalization of the holomorphic functional calculus. Let $S_L^{-1}(s,x)$ denote the *Cauchy kernel* for slice regular functions (cf. [3] or [8]).

Definition 5.2. [3, Def. 4.10.4] Let f be slice regular on $\Omega_D \supset \sigma_S(T)$. Fix any $\jmath \in \mathbb{S}$ and define

$$f(T)_{reg} := \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_\jmath)} S_L^{-1}(s,T)\,ds\,\jmath^{-1}\,f(s) \in \mathfrak{B}(V).$$

It can be shown that on slice regular functions our continuous calculus for normal operators on a quaternionic Hulbert space H coincides with the one defined by means of the Cauchy integral.

Proposition 5.3. *Let $T \in \mathfrak{B}(\mathsf{H})$ be normal and let $f : U \longrightarrow \mathbb{H}$ be a slice regular function defined on a circular open neighborhood of $\sigma_S(T)$ in $\mathbb{H}$. Then $f(T)_{reg} = f|_{\sigma_S(T)}(T)$, that is, the two functional calculi coincide if T is normal and f is slice regular.*

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