SLICE FUNCTION THEORY OF SEVERAL QUATERNIONIC VARIABLES
(IN A NUTSHELL)

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We recall some basic definitions from [1, 2]. Let D be a subset of $\mathbb{C}^n$ that is invariant under complex conjugations: $\overline{z}^h := (z_1, \dots, \overline{z}_h, \dots, z_n) \in D$ for all $z \in D$ and for all $h \in \{1, \dots, n\}$. Let $\{e_K\}_{K\in\mathcal{P}(n)}$ be a fixed basis of the real vector space $\mathbb{R}^{2^n}$. We identify $\mathbb{R}$ with the real vector subspace of $\mathbb{R}^{2^n}$ generated by $e_\emptyset \in \mathbb{R}^{2^n}$, and we write $e_\emptyset = 1$. For simplicity, we set $e_k := e_{\{k\}}$ for all $k \in \{1, \dots, n\}$.

Each element x of the tensor product $\mathbb{H} \otimes \mathbb{R}^{2^n}$ can be uniquely written as $x = \sum_{K\in\mathcal{P}(n)} e_K a_K$ with $a_K \in \mathbb{H}$. Given any function $F : D \to \mathbb{H} \otimes \mathbb{R}^{2^n}$, there exist unique functions $F_K : D \to \mathbb{H}$ such that $F = \sum_{K\in\mathcal{P}(n)} e_K F_K$ (F_K is called the **K-component of** F). A function $F : D \to \mathbb{H} \otimes \mathbb{R}^{2^n}$ with $F = \sum_{K\in\mathcal{P}(n)} e_K F_K$ is a **stem function** if

$$F_K(\overline{z}^h) = \begin{cases} F_K(z) & \text{if } h \notin K, \\ -F_K(z) & \text{if } h \in K \end{cases} \tag{1}$$

for all $z \in D$, $K \in \mathcal{P}(n)$ and $h \in \{1, \dots, n\}$. Let Ω_D be the **axially symmetric** (or **circular**) open subset of $\mathbb{H}^n$ associated to D, defined as

$$\Omega_D := \{(\alpha_1 + J_1\beta_1, \dots \alpha_n + J_n\beta_n) \in \mathbb{H}^n : J_1, \dots, J_n \in \mathbb{S}, (\alpha_1 + i\beta_1, \dots, \alpha_n + i\beta_n) \in D\}.$$

The **(left) slice function** $f = \mathcal{I}(F) : \Omega_D \to \mathbb{H}$ induced by F is the function obtained by setting, for each $x = (x_1, \dots, x_n) = (\alpha_1 + J_1\beta_1, \dots, \alpha_n + J_n\beta_n) \in \Omega_D$,

$$f(x) = \textstyle\sum_{K\in\mathcal{P}(n)} J_K F_K(z) \tag{2}$$

where $J_K = J_{k_1} \cdots J_{k_p}$ if $K = \{k_1, \dots, k_p\} \in \mathcal{P}(n) \setminus \{\emptyset\}$ with $k_1 < \cdots < k_p$, $J_\emptyset = 1$, and $z = (z_1, \dots, z_n) = (\alpha_1 + i\beta_1, \dots, \alpha_n + i\beta_n) \in D$.

We will denote by $\mathcal{S}^0(\Omega)$ the right $\mathbb{H}$-module of slice functions in $\Omega = \Omega_D$ induced by continuous stem functions $F \in C^0(D)$, and by $\mathcal{S}^1(\Omega)$ the submodule of slice functions in $\Omega = \Omega_D$ induced by stem functions $F \in C^1(D)$.

Let $\mathcal{J}_1, \dots \mathcal{J}_n$ be the commuting complex structures on $\mathbb{R}^{2n} \simeq \mathbb{C}^{\otimes n}$ induced, respectively, by the standard structures of the n copies of $\mathbb{C}$. The isomorphism $\mathbb{R}^{2n} \simeq \mathbb{C}^{\otimes n}$ maps the basis element e_K of $\mathbb{R}^{2n}$ to the element v_K of $\mathbb{C}^{\otimes n}$ defined as

$$v_K = v_1 \otimes \cdots \otimes v_n \quad \text{with } v_h = i \text{ if } h \in K,\ v_h = 1 \text{ if } h \notin K.$$

Explicitly, the complex structures are defined by

$$\mathcal{J}_h(e_K) = \begin{cases} e_{K\cup\{h\}} & \text{if } h \notin K, \\ -e_{K\setminus\{h\}} & \text{if } h \in K. \end{cases} \tag{3}$$

In particular, it holds $\mathcal{J}_h(e_h) = -1$ for $h = 1, \dots, n$. We extend these structures to $\mathbb{H} \otimes \mathbb{R}^{2n}$ by setting $\mathcal{J}_h(a \otimes v) = a \otimes \mathcal{J}_h(v)$ for all $a \in \mathbb{H}$ and $v \in \mathbb{R}^{2n}$.

Let $F : D \to \mathbb{H} \otimes \mathbb{R}^{2n}$ be a stem function of class $\mathscr{C}^1$. For each $h = 1, \dots, n$, we denote by ∂_h and $\overline{\partial}_h$ the **Cauchy-Riemann operators** w.r.t. the standard complex structure on D and $\mathcal{J}_h$ on $\mathbb{H} \otimes \mathbb{R}^{2n}$, i.e.

$$\partial_h F = \frac{1}{2}\left(\frac{\partial F}{\partial \alpha_h} - \mathcal{J}_h\left(\frac{\partial F}{\partial \beta_h}\right)\right) \quad \text{and} \quad \overline{\partial}_h F = \frac{1}{2}\left(\frac{\partial F}{\partial \alpha_h} + \mathcal{J}_h\left(\frac{\partial F}{\partial \beta_h}\right)\right), \tag{4}$$

where $\alpha_h + i\beta_h : D \to \mathbb{C}$ is the h^{th}-coordinate function of D. Let $f = \mathcal{I}(F) : \Omega_D \to \mathbb{H}$ and let $h \in \{1, \dots, n\}$. Each operator of the type ∂_h or $\overline{\partial}_h$ commutes with each other. We define the **slice partial derivatives** of f as the following slice functions on Ω_D:

$$\boxed{\frac{\partial f}{\partial x_h} = \mathcal{I}(\partial_h F) \quad \text{and} \quad \frac{\partial f}{\partial x_h^c} = \mathcal{I}(\overline{\partial}_h F).} \tag{5}$$

The slice function $f = \mathcal{I}(F)$ is called **slice-regular** on Ω_D if F is holomorphic w.r.t. $\mathcal{J}_1, \dots, \mathcal{J}_n$, i.e., $\overline{\partial}_h F = 0$ for $h \in \{1, \dots, n\}$. Equivalently, $\frac{\partial f}{\partial x_h^c} = 0$ for every h. For example, every polynomial function $p(x) = \sum_\ell x_1^{\ell_1} \cdots x_n^{\ell_n} a_\ell$ with ordered variables and right quaternionic coefficients a_ℓ, with $\ell = (\ell_1, \dots, \ell_n)$, is slice-regular on $\mathbb{H}^n$. We will denote by $\mathcal{SR}(\Omega)$ the right quaternionic module of slice-regular functions on $\Omega = \Omega_D$.

Every product on $\mathbb{H} \otimes \mathbb{R}^{2n}$ induces a product on stem functions, and hence a structure of real algebra on the set of slice functions. In the following we take the product obtained identifying as above the real algebra $\mathbb{R}^{2n}$ with $\mathbb{C}^{\otimes n}$. The corresponding (tensor) product in $\mathbb{H} \otimes \mathbb{R}^{2n}$ is the linear extension of

$$(a \otimes e_H)(b \otimes e_K) = (ab) \otimes (e_H e_K) = (ab)(-1)^{|H \cap K|} e_{H \Delta K}.$$

Let $f = \mathcal{I}(F), g = \mathcal{I}(G) : \Omega_D \to \mathbb{H}$ be slice functions. We define the **slice product** $f \cdot g : \Omega_D \to \mathbb{H}$ of f and g by $f \cdot g := \mathcal{I}(FG)$, where FG is the pointwise product defined by $(FG)(z) = F(z)G(z)$ in $\mathbb{H} \otimes \mathbb{R}^{2n}$ for all $z \in D$.

The slice partial derivatives satisfy Leibniz's rule w.r.t. the slice product: for each slice functions $f, g \in \mathcal{S}^1(\Omega_D)$ and $h = 1, \dots, n$, it holds

$$\frac{\partial}{\partial x_h}(f \cdot g) = \frac{\partial f}{\partial x_h} \cdot g + f \cdot \frac{\partial g}{\partial x_h} \quad \text{and} \quad \frac{\partial}{\partial x_h^c}(f \cdot g) = \frac{\partial f}{\partial x_h^c} \cdot g + f \cdot \frac{\partial g}{\partial x_h^c}. \tag{6}$$

In particular, the slice product preserves slice regularity.

One-variable characterization. The concepts of spherical value and spherical derivative in one variable have a central role in the characterization of slice regularity in several variables in terms of separate one-variable regularity. Assume that g is a slice function w.r.t. x_h and define the functions $\mathcal{D}^0_{x_h} g(x)$ and $\mathcal{D}^1_{x_h} g(x)$ obtained taking the spherical value and the spherical derivative of g w.r.t. x_h:

$$\mathcal{D}^0_{x_h} g := (g)^\circ_{s,x_h} \quad \text{and} \quad \mathcal{D}^1_{x_h} g := (g)'_{s,x_h}. \tag{7}$$

Let $f \in \mathcal{S}(\Omega)$ be a slice function on $\Omega \subseteq \mathbb{H}^n$, let $K \in \mathcal{P}(n)$ and let $\epsilon = \mathbf{1}_K$ be the characteristic function of K. Then f is a slice function w.r.t. x_1 and, for each $h \in \{2, \dots, n\}$, the **truncated spherical ϵ-derivative** $\mathcal{D}_\epsilon f := \mathcal{D}^{\epsilon(h-1)}_{x_{h-1}} \cdots \mathcal{D}^{\epsilon(1)}_{x_1} f$, obtained iterating (7), is a well-defined slice function w.r.t. x_h. Moreover, $f \in \mathcal{SR}(\Omega)$ if and only if f is slice-regular w.r.t. x_1 and, for each $h \in \{2, \dots, n\}$ and $K \in \mathcal{P}(n)$, $\mathcal{D}_\epsilon f$ is slice-regular w.r.t. x_h [2, Proposition 2.23 and Theorem 3.23].

For example, when $n = 2$, a slice function f is slice-regular in $x = (x_1, x_2)$ if and only if f is slice-regular w.r.t. x_1, and the spherical value and spherical derivative of f w.r.t. x_1 are slice-regular w.r.t. x_2.

References

[1] R. Ghiloni and A. Perotti. Slice regular functions of several Clifford variables. In *AIP Conf. Proc.*, volume 1493, pages 734–738. AIP, 2012. 9th International Conference on Mathematical Problems in Engineering, Aerospace and Sciences: ICNPAA 2012, Vienna, Austria, 10–14 July 2012.

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