MULTIDIMENSIONAL RESIDUES AND IDEAL MEMBERSHIP

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Abstract. Let $I(f)$ be a zero-dimensional ideal in $\mathbf{C}[z_1,\dots,z_n]$ defined by a mapping f. We compute the logarithmic residue of a polynomial g with respect to f. We adapt an idea introduced by Aizenberg to reduce the computation to a special case by means of a limiting process.

We then consider the total sum of local residues of g w.r.t. f. If the zeroes of f are simple, this sum can be computed from a finite number of logarithmic residues. In the general case, you have to perturb the mapping f.

Some applications are given. In particular, the global residue gives, for any polynomial, a canonical representative in the quotient space $\mathbf{C}[z]/I(f)$.

1. Introduction

We present some algebraic applications of the theory of multidimensional residues in $\mathbf{C}^n$. The logarithmic residues and the local (or Grothendieck) residues have been studied by many authors. In particular, we consider some ideas of Aizenberg, Tsikh and Yuzhakov (see [3] or [6] for a survey).

Let $I(f)$ be a zero-dimensional ideal in $\mathbf{C}[z_1,\dots,z_n]$ defined by a polynomial mapping f. In Section 2 we consider the problem of computing the logarithmic residue of a polynomial g with respect to f. In the special case when the principal part of every component f_i is a power $z_i^{k_i}$, we give a method in order to simplify the computation. We reduce it to the application, to only one special polynomial, of a linear functional introduced by Aizenberg [1] and to the finding of the projection of g onto a finite-dimensional subspace of $\mathbf{C}[z_1,\dots,z_n]$. We also give a description of the radical of I.

In the general case, we adapt an idea introduced by Aizenberg to reduce to the special case by means of a limiting process (Proposition 2 and Theorem 1).

In Section 3 we consider the total sum of local residues of a polynomial with respect to the mapping f. If all the zeroes of f are simple, we show that this sum can be computed from a finite number of logarithmic residues. In the general case, you have to perturb the mapping f to get a similar result (Theorem 2).

In Section 4 we say something about the applications of these results. In particular, we show (Proposition 3) how the total sum of residues gives, for any polynomial, a canonical representative of its class in the quotient space $\mathbf{C}[z]/I(f)$.

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2. Logarithmic Residues

2.1. Let $I = I(f) = (f_1, \dots, f_n)$ be a zero-dimensional polynomial ideal in $\mathbf{C}[z] = \mathbf{C}[z_1, \dots, z_n]$. This means that the zero set $V(f) = V(f_1, \dots, f_n)$ is a discrete algebraic variety in $\mathbf{C}^n$, with at most $\deg(f_1) \cdots \deg(f_n)$ points, counted with their multiplicities. Let $z^{(1)}, \dots, z^{(N)}$ be these (possibly repeated) points. Given a polynomial $g \in \mathbf{C}[z]$, we want to compute the logarithmic residue of g with respect to the mapping $f = (f_1, \dots, f_n)$, that is the sum

$$\mathrm{LRes}_f(g) = \sum_{\nu=1}^{N} g(z^{(\nu)})$$

2.2. We first consider the special case when $f_i = z_i^{k_i} + P_i$, $i = 1, \dots, n$, where the total degree of P_i is less than k_i. In this situation, the logarithmic residue is given by an explicit formula introduced by Aizenberg (see [1], [3], [4], [6]), which can be derived from the application of the Leray-Koppelman integral representation formula for holomorphic functions (see for example [4] Section 3) on a pseudoball in $\mathbf{C}^n$:

$$\mathrm{LRes}_f(g) = \mathcal{N}\left(gJ \frac{z_1 \cdots z_n}{z_1^{k_1} \cdots z_n^{k_n}} \sum_{|\alpha|=0}^{\deg(g)} (-1)^{|\alpha|} \left(\frac{P_1}{z_1^{k_1}}\right)^{\alpha_1} \cdots \left(\frac{P_n}{z_n^{k_n}}\right)^{\alpha_n} \right)$$

where J is the Jacobian determinant of the mapping f and $\mathcal{N}$ is the linear functional on the polynomials in $z_1, \dots, z_n$ and $1/z_1, \dots, 1/z_n$ that assigns to each polynomial its free term.

We show that the computation of $\mathrm{LRes}_f(g)$ can be simplified by exploiting the decomposition $\mathbf{C}[z] = \mathbf{C}_{k-1}[z] \oplus I$, where $\mathbf{C}_{k-1}[z]$ is the N-dimensional space of the polynomials in $\mathbf{C}[z]$ with degree less than k_i with respect to z_i for every $i = 1, \dots, n$. This follows from the particular form of the polynomials f_i. In fact, it can be easily seen that $f_1, \dots, f_n$ is a Gröbner basis (not necessarily reduced) of the ideal I with respect to any degree ordering.

Let z^α denote the monomial $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. Let $K_0(z, \zeta) \in \mathbf{C}[z, \zeta]$ be a polynomial which belongs to $\mathbf{C}_{k-1}[z]$ for any fixed ζ and to $\mathbf{C}_{k-1}[\zeta]$ for any fixed z and has the following property:

(*) the set $\{K_\alpha(\zeta)\}$ defined by the decomposition $K_0(z, \zeta) = \sum_\alpha K_\alpha(\zeta) z^\alpha$ is a basis of $\mathbf{C}_{k-1}[\zeta]$.

Let $K(z) = \mathrm{LRes}_f(K_0) = \sum_\alpha \mathrm{LRes}_f(K_\alpha) z^\alpha$. Consider the non-degenerate bilinear form on $\mathbf{C}[z]$ defined for any $p = \sum_\alpha a_\alpha z^\alpha$, $q = \sum_\alpha b_\alpha z^\alpha$ by

$$\langle p, q \rangle_K = \sum_{\alpha,\beta} m_{\alpha,\beta} a_\alpha b_\beta$$

where $M = (m_{\alpha,\beta})$ is the transition matrix from the basis $\{K_\alpha\}$ to the basis $\{z^\beta\}_{0 \le \beta_i < k_i}$.

Then we get the following result.

Proposition 1. *The logarithmic residue of $g \in \mathbf{C}[z]$ with respect to f is given by the linear functional $\langle \cdot , K \rangle_K$ evaluated on the (unique) projection g_0 of g in $\mathbf{C}_{k-1}[z]$.*

Proof. if $g = g_0 + g_1 \in \mathbf{C}_{k-1}[z] \oplus I$ and $g_0 = \sum_\alpha a_\alpha z^\alpha = \sum_{\alpha,\beta} m_{\alpha,\beta} a_\alpha K_\beta$, then $\mathrm{LRes}_f(g) = \mathrm{LRes}_f(g_0) = \sum_{\alpha,\beta} m_{\alpha,\beta} a_\alpha \mathrm{LRes}_f(K_\beta) = \langle g_0, K \rangle_K$.

Two possible choices for the kernel $K_0(z,\zeta)$ are the following:

(i) $K_0(z,\zeta) = \sum_{0\le\alpha_i<k_i}\prod_i (z_i\zeta_i)^{\alpha_i}$, with associated form $\langle p,q\rangle = \sum_\alpha a_\alpha b_\alpha$;

(ii) $K_0(z,\zeta) = \prod_i \dfrac{(\zeta_i^{k_i} - z_i^{k_i})}{(\zeta_i - z_i)}$, with associated form $\langle p,q\rangle = \sum_\alpha a_\alpha b_{k-\alpha-1}$, where $k-\alpha-1$ is the multiindex $(k_1-\alpha_1-1,\dots,k_n-\alpha_n-1)$.

Remark. The second kernel is a Hefer determinant of the mapping $Q = f - P = (z_1^{k_1},\dots,z_n^{k_n})$. It is the determinant of the polynomial matrix $(P_{ij}(z,\zeta))$ defined by the Hefer expansions

$$Q_i(\zeta) - Q_i(z) = \sum_j P_{ij}(z,\zeta)(\zeta_j - z_j)$$

Remark. If K_0 have integer coefficients, then the coefficients of $K(z)$ are integer polynomial expressions in the coefficients of the f_i. If the f_i have integer, rational or real coefficients respectively, the same holds for $K(z)$.

2.3. Let $K_0(z,\zeta)$ be the kernel given in (i). If the polynomials f_i have real coefficients, then $\langle K,K\rangle_K$ is a real number greater than N^2, since $K(0) = N$. It follows the decomposition $\mathbf{C}[z] = \langle K\rangle \oplus \left(\mathbf{C}_{k-1}[z]\cap\langle K\rangle^\perp\right)\oplus I$, where the second subspace is formed by the polynomials $g\in\mathbf{C}_{k-1}[z]$ such that $\mathrm{LRes}_f(g) = 0$. Then the set of polynomials vanishing on $V(f)$, that is the radical ideal $\mathrm{Rad}\, I$, decomposes as

$$\mathrm{Rad}\, I = (\mathrm{Rad}\, I\cap\mathbf{C}_{k-1}[z])\oplus I$$

with

$$\mathrm{Rad}\, I\cap\mathbf{C}_{k-1}[z] = \left\{g\in\langle K\rangle^\perp\cap\mathbf{C}_{k-1}[z]\ :\ (g^l)_0\in\langle K\rangle^\perp \text{ for every } l = 2,\dots,N\right\}$$

Here $(g^l)_0$ denotes the component of g^l in $\mathbf{C}_{k-1}[z]$.

Remark. Since $\langle K_0(a,\zeta),K(\zeta)\rangle_K = K(a)$, if K is not the constant N we get that $K_0(a,\zeta)\in\langle K\rangle^\perp\cap\mathbf{C}_{k-1}[\zeta]$ if and only if $K(a) = 0$.

2.4. Now we return to the general case. Let $f = (f_1,\dots,f_n)$ be a polynomial mapping with a discrete zero set $V(f) = \{z^{(1)},\dots,z^{(N)}\}$. Let $k_i = \deg(f_i)$ for $i = 1,\dots,n$. Then $N\le k_1\cdots k_n$.

We use an idea introduced by Aizenberg to reduce the general case to the previous case.

If, for some i, the polynomial f_i has the special form considered in section 2.1, with principal part $z_j^{k_i}$, we set $f'_j = f_i$. For the remaining indices, we set $f'_i = z_i^{k_i+1} + \mu f_i$, $\mu\in\mathbf{C}$. Let I'_μ be the ideal generated by $f'_1,\dots,f'_n$. It has zero set $V(f')$ containing $M = \deg(f'_1)\cdots\deg(f'_n)$ points (with multiplicities), which we shall denote by $z_\mu^{(1)},\dots,z_\mu^{(M)}$. If f is not in the special form, than $M > N$.

Let $g\in\mathbf{C}[z]$. Let $a = (a_1,\dots,a_n)$ be a vector of complex parameters and $g' = g + \sum_i a_i z_i$. For any fixed value of μ, f' has the special form considered in 2.1. Then we can compute the logarithmic residues $\mathrm{LRes}_{f'}((g')^l)$, $l = 1\dots,M$. These are polynomial expressions in $\mu,a_1,\dots,a_n$. From Newton's formula, we can find the elementary symmetric functions $\sigma_{g'}^l(\mu)$ in the quantities $g'(z_\mu^{(1)}),\dots,g'(z_\mu^{(M)})$.

It follows from Rouché's principle (see [4] Section 2) that N elements of $V(f')$ tend to the points in $V(f)$ as $\mu \to \infty$, while the other $M - N$ points tend to ∞. After reordering, we can assume that $z_\mu^{(1)}, \ldots, z_\mu^{(N)}$ have limits $z^{(1)}, \ldots, z^{(N)}$ respectively.

Let us denote by $\sigma_{g'}^l$, $l = 1, \ldots, N$, the elementary symmetric functions in $g'(z^{(1)}), \ldots, g'(z^{(N)})$. The polynomial g' can vanish identically (with respect to a) only in the point 0 and in this case $g(0) = 0$. If $0 \in V(f)$, then $0 \in V(f')$ with the same multiplicity h. Assume that $z_\mu^{(1)} = 0, \ldots, z_\mu^{(h)} = 0$. Let us denote by $\sigma_{g'}^{-l}$, $l = 1, \ldots, N - h$, the elementary symmetric functions in $g'(z^{(h+1)})^{-1}, \ldots, g'(z^{(N)})^{-1}$.

Proposition 2. (i) $\sigma_g^l = \lim_{a \to 0} \sigma_{g'}^l$ *for every* $l = 1, \ldots, N$*;*

(ii) $\sigma_{g'}^l = \lim_{\mu \to \infty} \dfrac{\sigma_{g'}^{M-N+l}(\mu)}{\sigma_{g'}^{M-N}(\mu)}$ *for every* $l = 1, \ldots, N$.

Proof. (i) is immediate, since $\sigma_{g'}^l$ depends polynomially from a; for (ii), we adapt the arguments given in [4] (Section 21.3). If $0 \notin V(f)$ then $\sigma_{g'}^M(\mu) \not\equiv 0$. For all a with the exception of a set of complex dimension $n - 1$, the ratios $\sigma_{g'}^{M-l}(\mu)\big(\sigma_{g'}^M(\mu)\big)^{-1}$ tend to 0 for $l = N + 1, \ldots, M$, and to $\sigma_{g'}^{-l}$ for $l = 1, \ldots, N$. But the functions $\sigma_{g'}^l(\mu)$ are polynomials in $\mathbf{C}(a)[\mu]$ and therefore the ratios $\sigma_{g'}^{M-l}(\mu)\big(\sigma_{g'}^M(\mu)\big)^{-1}$ have limit in $\mathbf{C}(a)$, as $\mu \to \infty$, equal to 0 for $l = N + 1, \ldots, M$, and equal to $\sigma_{g'}^{-l}$ for $l = 1, \ldots, N$.

Then $\sigma_{g'}^{M-N+l}(\mu)\Big(\sigma_{g'}^{M-N}(\mu)\Big)^{-1}$ tends to $\sigma_{g'}^{-N+l}\Big(\sigma_{g'}^{-N}\Big)^{-1} = \sigma_{g'}^l$ for every $l = 1, \ldots, N$.

If $0 \in V(f)$ with multiplicity h, then $\sigma_{g'}^l(\mu) \equiv 0$ for $l = M - h + 1, \ldots, M$, while $\sigma_{g'}^{M-h}(\mu) \not\equiv 0$. The ratios $\sigma_{g'}^{M-h-l}(\mu)\Big(\sigma_{g'}^{M-h}(\mu)\Big)^{-1}$ tend to 0 for $l = N-h+1, \ldots, M-h$, and to $\sigma_{g'}^{-l}$ for $l = 1, \ldots, N-h$. In particular, $\sigma_{g'}^{M-N}(\mu)\Big(\sigma_{g'}^{M-h}(\mu)\Big)^{-1}$ has limit $\sigma_{g'}^{-N+h} \not\equiv 0$, hence $\sigma_{g'}^{M-N}(\mu) \not\equiv 0$.

It remains to note that $\sigma_{g'}^l = \sigma_{g'}^{-N+h+l}\Big(\sigma_{g'}^{-N+h}\Big)^{-1}$ for every $l = 1, \ldots, N - h$.

Remark. In general, the number N is not known in advance. It can be determined from the previous limiting processes, by counting how many ratios $\sigma_{g'}^{M-h-l}(\mu)\Big(\sigma_{g'}^{M-h}(\mu)\Big)^{-1}$ tend to 0. Equivalently, it is the number of functions $\sigma_{g'}^{M-h-l}(\mu)$ which have the same μ-degree as $\sigma_{g'}^{M-h}(\mu)$.

In particular, $\sigma_g^1 = \mathrm{LRes}_f(g)$. We have proved the following result.

Theorem 1. *The logarithmic residue of any* $g \in \mathbf{C}[z]$ *with respect to* f *can be computed from*

$$\mathrm{LRes}_f(g) = \lim_{a \to 0} \lim_{\mu \to \infty} \frac{\sigma_{g'}^{M-N+1}(\mu)}{\sigma_{g'}^{M-N}(\mu)}$$

3. Local Residues

Now we consider the total sum of local residues of a polynomial $g \in \mathbf{C}[z]$ with respect to the polynomial mapping $f = (f_1, \ldots, f_n)$. In general, if $f = (f_1, \ldots, f_n)$

is a holomorphic mapping with an isolated zero a in a closed neighbourhood U_a of a, the local (or Grothendieck) residue at a of a holomorphic function g on U_a with respect to f is the integral

$$\mathrm{res}_{a,f}(g) = \frac{1}{(2\pi i)^n} \int_{\Gamma_a(f)} \frac{g\ dz_1 \wedge \cdots \wedge dz_n}{f_1 \cdots f_n}$$

where $\Gamma_a(f)$ is the n-chain $= \{z \in U_a : |f_i(z)| = \epsilon_i,\ i = 1, \ldots, n\}$, with $\epsilon_i > 0$ such that $\Gamma_a(f)$ is relatively compact in U_a (see for example [5]).

3.1. Let $I = (f_1, \ldots, f_n)$ be a zero-dimensional polynomial ideal in $\mathbf{C}[z]$. Since f has a finite number of isolated zeroes, we can consider the global residue $\mathrm{Res}_f(g) = \sum_{a \in V(f)} \mathrm{res}_{a,f}(g)$ of the local residues of $g \in \mathbf{C}[z]$ with respect to f.

Remark. If $g = h \cdot J$, where J is the Jacobian determinant of the mapping f, the local residue coincides with the logarithmic residue of h at a. Then $\mathrm{LRes}_f(h) = \mathrm{Res}_f(h \cdot J)$.

If f has the special form $f_i = z_i^{k_i} + P_i$, with $\deg(P_i) < k_i$, the global residue $\mathrm{Res}_f(g)$ can be computed from the explicit formula of Aizenberg [1].

In the general case, Yuzhakov introduced in [9] an algorithm to reduce the problem to the special case, by applying the transformation formula for the local residue and the generalized resultants. We proceed in a different way. We obtain $\mathrm{Res}_f(g)$ from the computation of a finite number of (global) logarithmic residues, which can be found with the method of Section 2.

3.2. In the case that the zeroes $z^{(1)}, \ldots, z^{(N)}$ of f are all simple, then $\mathrm{Res}_f(g) = \sum_{\nu=1}^{N} \dfrac{g(z^{(\nu)})}{J(z^{(\nu)})}$. We can now apply the following lemma, which generalizes Newton's formulas (for a proof, see for example [7]).

Lemma 1. *Let $\sigma^l(a)$ denote the l-th elementary symmetric function of m scalars $a_1, \ldots, a_m$. If $b_1, \ldots, b_m$ are scalars different from zero, the sum $\sigma^1\left(\frac{a}{b}\right) = \frac{a_1}{b_1} + \cdots + \frac{a_m}{b_m}$ is given by*

$$\sigma^1\left(\frac{a}{b}\right) = \sum_{k=0}^{m-1} (-1)^k \frac{\sigma^1(ab^k) \cdot \sigma^{m-k-1}(b)}{\sigma^m(b)}$$

Then we obtain the following formula:

$$\mathrm{Res}_f(g) = \sum_{k=0}^{N-1} (-1)^k \frac{\sigma^1_{g \cdot J^k} \cdot \sigma_J^{N-k-1}}{\sigma_J^N}$$

where $\sigma^1_{g \cdot J^k}$ and σ^l_J can be found from Proposition 2.

3.3. If not all the zeroes of f are simple, f can be perturbed. We consider $f - w$, where w is a small complex n-tuple. For generic values of w, the Jacobian J does not vanish at the zeroes of $f - w$. Let $z^{(1)}(w), \ldots, z^{(N)}(w)$ be the elements of $V(f - w)$. In [6], Section 6.2, Tsikh showed that the sum

$$\phi(w) = \sum_{\nu=1}^{N} \frac{g(z^{(\nu)}(w))}{J(z^{(\nu)}(w))}$$

is a holomorphic function in w on a small neighbourhood of 0. Then $\phi(0)$ is the sum of the local residues of g at the zeroes of f. As a result, we obtain the following theorem.

Theorem 2. *The global residue* $\operatorname{Res}_f(g)$ *of any* $g \in \mathbf{C}[z]$ *with respect to* f *is equal to* $\psi(0)$*, where* $\psi(w)$ *is the holomorphic function given by*

$$\psi(w) = \sum_{k=0}^{N-1} (-1)^k \frac{\operatorname{LRes}_{f-w}(g \cdot J^k) \cdot \sigma_J^{N-k-1}(w)}{\sigma_J^N(w)}$$

Here $\sigma^l_J(w)$ are the elementary symmetric functions in $J(z^{(1)}(w)),\dots,J(z^{(N)}(w))$, which can be found from the logarithmic residues $\operatorname{LRes}_{f-w}(J^l)$, $l = 1,\dots,N$.

4. Applications

4.1. The global residues and the total logarithmic residues have well known applications. They give a method for eliminating variables which does not use resultants. For any $i = 1,\dots,n$, from LRes_f a univariate polynomial in $I(f) \cap \mathbf{C}[z_i]$ of degree N can be computed. It preserves multiplicities of the zeroes of f (for this method, see [4] Section 21).

From Res_f a membership criterion for the ideal $I(f)$ can be deduced. In [8], Tsikh applied Lasker-Noether Theorem and got the following:

$$g \in I(f) \Leftrightarrow \operatorname{Res}_f(g(\zeta)H(z,\zeta)) = 0, \text{ where } H(z,\zeta) \text{ is a Hefer determinant of } f$$

Remark. A polynomial Hefer determinant of f can be computed from the Hefer expansions

$$f_i(\zeta) - f_i(z) = \sum_j P_{ij}(z,\zeta)(\zeta_j - z_j)$$

where $P_{ij}(z,\zeta) = \dfrac{f_i(\zeta_1,\dots,\zeta_j,z_{j+1},\dots,z_n) - f_i(\zeta_1,\dots,\zeta_{j-1},z_j,\dots,z_n)}{\zeta_j - z_j}$.

Note that from $P_{ij}(z,\zeta)$ and $P_{ij}(\zeta,z)$ we can get a Hefer determinant which is symmetric in z and ζ.

4.2. Let $g,h \in \mathbf{C}[z]$ and $g_0(\zeta) = \operatorname{Res}_f(g(z)H(z,\zeta))$, $h_0(\zeta) = \operatorname{Res}_f(h(z)H(z,\zeta))$. From the membership criterion above we get that $g_0 = h_0$ if and only if the difference $g - h \in I(f)$, that is g and h define the same class in the N-dimensional quotient space $\mathbf{C}[z]/I(f)$.

If we apply the transformation formula for the global residue (see [8]) to the Hefer expansion of f, we get, for any polynomial p, $\operatorname{Res}_{z-\zeta} p(z) = \operatorname{Res}_{f-f(\zeta)}(p(z)H(z,\zeta))$. It follows that for any $a \in V(f)$, $\operatorname{Res}_f(p(z)H(z,a)) = p(a)$. In particular, we get $\operatorname{Res}_f H(z,a) = 1$.

From this we can deduce that $\operatorname{Res}_f(g_0(z)H(z,\zeta)) = \operatorname{Res}_f(g(z)H(z,\zeta)) = g_0(\zeta)$. For simplicity, assume that the zeroes of f are simple. Then

$$\begin{aligned}
\operatorname{Res}_f(g_0(z)H(z,\zeta)) &= \sum_\nu \frac{g_0(z^\nu)H(z^\nu,\zeta)}{J(z^\nu)} \\
&= \sum_{\nu,\mu} \frac{g(z^\mu)H(z^\nu,z^\mu)H(z^\nu,\zeta)}{J(z^\nu)J(z^\mu)} \\
&= \sum_\mu \frac{g(z^\mu)}{J(z^\mu)} \operatorname{Res}_f(H(z,z^\mu)H(z,\zeta)) \\
&= \operatorname{Res}_f(g(z)H(z,\zeta)) = g_0(\zeta).
\end{aligned}$$

As a result, we get the following proposition.

Proposition 3. *Let* $g \in \mathbf{C}[z]$, $g_0(\zeta) = \mathrm{Res}_f(g(z)H(z,\zeta))$. *Then* $g - g_0 \in I(f)$, *that is* g_0 *represents* g *in the quotient space* $\mathbf{C}[z]/I(f)$. *In particular,* $\mathrm{Res}_f\, g = \mathrm{Res}_f\, g_0$.

Note added in proof. The paper [E. Cattani, A. Dickenstein, B. Sturmfels, Computing multidimensional residues, Progress in Mathematics, Vol. 143, Birkhäuser Verlag, Basel, 1996, pp. 135–164] contains interesting relations between global residues and Gröbner bases.

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